Numerical implementation of Manzari-Dafalias model in $\mathbb{R}^3$ and $\mathbb{R}^6$ space, a computational comparison

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-CeiBA-

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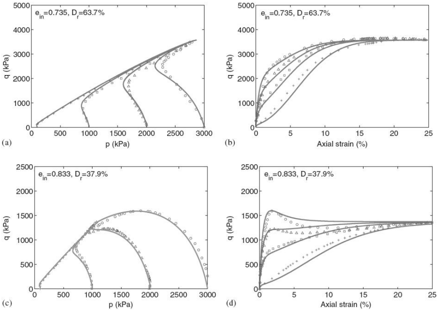
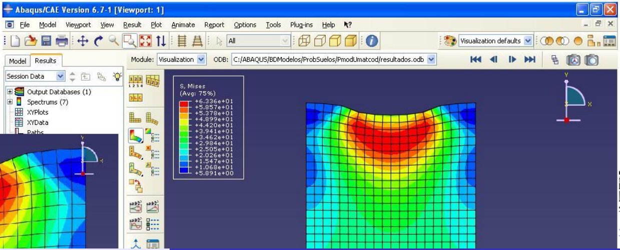
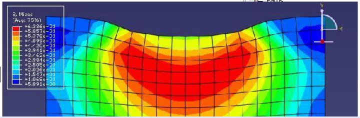
Motivation

Toyura Sand - Dafalias (Sanisand, 2007), and New building in Civil Eng. Faculty, Univ. of the Cauca





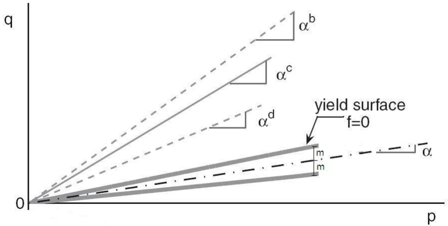


Outline

Yield surface in space p-q, Lawrence(1972)

$$f = \eta - \alpha \pm m = 0, f = q - p\alpha \pm mp = 0$$

α: Slope in $q - p$ space of the bisector of the ysurf.
$2m$: wedge 'opening' of the yield surf (angle). ($m = 0{,}05$)



The models developed over Elastic, hypoelastic and plastic theory, Critical state soil mechanics and Bounding surface plasticity

Is adopte

$$\dot{\varepsilon}_v^e = \frac{\dot{p}}{K} \qquad (1)$$

$$\dot{\boldsymbol{e}}^e = \frac{\dot{\boldsymbol{s}}}{2G} \qquad (2)$$

$$K = K_0 \left(\frac{p}{p_{atm}} \right)^a \qquad (3)$$

$$G = G_0 \left(\frac{p}{p_{atm}} \right)^a \qquad (4)$$

where $K_0, G_0, a(a \circlearrowleft 0{,}5)$: model parameter

p_{atm}: Atmospheric pressure

Yield surface

$$f = [(\boldsymbol{s} - p\boldsymbol{\alpha}) : (\boldsymbol{s} - p\boldsymbol{\alpha})]^{1/2} - \sqrt{\frac{2}{3}} mp = 0$$

$$f = [(\boldsymbol{r} - \boldsymbol{\alpha}) : (\boldsymbol{r} - \boldsymbol{\alpha})]^{1/2} - \sqrt{\frac{2}{3}} m = 0$$

back-stress ratio deviatoric tensor, $\boldsymbol{r} = \frac{\boldsymbol{s}}{p}$

Since consistency condition ($\dot{f} = 0$), then:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \boldsymbol{L} = \boldsymbol{n} - \frac{1}{3} N \boldsymbol{I} \tag{5}$$

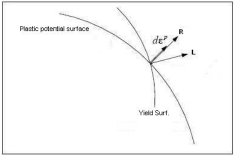
where:

$$\boldsymbol{n} = \frac{\boldsymbol{r} - \boldsymbol{\alpha}}{\sqrt{\frac{2}{3}} m} \tag{6}$$

$$N = \boldsymbol{\alpha} : \boldsymbol{n} + \sqrt{\frac{2}{3}} m \tag{7}$$

and:

$\boldsymbol{L}$: Loading direction



$$\frac{\partial g}{\partial \boldsymbol{\sigma}} = \boldsymbol{R} = \boldsymbol{n} - \frac{1}{3} D \boldsymbol{I} \tag{8}$$

Flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \langle L \rangle \boldsymbol{R} \tag{9}$$

where, D: Dilatancy coefficient, L: Loading index

The strain tensor in covariant base in $\mathbb{R}^6$

$$[\bar{\varepsilon}_a] = \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix}^T = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ (\varepsilon_{12} + \varepsilon_{21}) \\ (\varepsilon_{23} + \varepsilon_{32}) \\ (\varepsilon_{31} + \varepsilon_{13}) \end{bmatrix}^T = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix}^T \tag{10}$$

$$\begin{aligned} \mathbf{G}_1 &= \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1, & \mathbf{G}_4 &= \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 = 2\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2, \\ \mathbf{G}_2 &= \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2, & \mathbf{G}_5 &= \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 = 2\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2, \\ \mathbf{G}_3 &= \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3, & \mathbf{G}_6 &= \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 = 2\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \end{aligned} \tag{11}$$

$$[G_{ab}] = [G_a : G_b] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{12}$$

The stress tensor in contravariant base in $\mathbb{R}^6$

$$[G^{ab}] = [G_{ab}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \tag{13}$$

$$\begin{aligned} \mathbf{G}^1 &= \hat{\mathbf{e}}^1 \otimes \hat{\mathbf{e}}^1, & \mathbf{G}^4 &= \frac{1}{2}(\hat{\mathbf{e}}^1 \otimes \hat{\mathbf{e}}^2 + \hat{\mathbf{e}}^2 \otimes \hat{\mathbf{e}}^1) = \hat{\mathbf{e}}^1 \otimes \hat{\mathbf{e}}^2, \\ \mathbf{G}^2 &= \hat{\mathbf{e}}^2 \otimes \hat{\mathbf{e}}^2, & \mathbf{G}^5 &= \frac{1}{2}(\hat{\mathbf{e}}^2 \otimes \hat{\mathbf{e}}^3 + \hat{\mathbf{e}}^3 \otimes \hat{\mathbf{e}}^2) = \hat{\mathbf{e}}^2 \otimes \hat{\mathbf{e}}^3, \\ \mathbf{G}^3 &= \hat{\mathbf{e}}^3 \otimes \hat{\mathbf{e}}^3, & \mathbf{G}^6 &= \frac{1}{2}(\hat{\mathbf{e}}^3 \otimes \hat{\mathbf{e}}^1 + \hat{\mathbf{e}}^1 \otimes \hat{\mathbf{e}}^3) = \hat{\mathbf{e}}^1 \otimes \hat{\mathbf{e}}^3 \end{aligned} \tag{14}$$

$$[\bar{\sigma}^a] = \begin{bmatrix} \bar{\sigma}^1 \\ \bar{\sigma}^2 \\ \bar{\sigma}^3 \\ \bar{\sigma}^4 \\ \bar{\sigma}^5 \\ \bar{\sigma}^6 \end{bmatrix} = \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ 1/2(\sigma^{12} + \sigma^{21}) \\ 1/2(\sigma^{23} + \sigma^{32}) \\ 1/2(\sigma^{31} + \sigma^{13}) \end{bmatrix} = \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{23} \\ \sigma^{31} \end{bmatrix} \tag{15}$$

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Elastic Tensor

For each index $a = \{1, 2, 3, 4, 5, 6\}$ exist a pair $\{i, j\} = \{\{1, 1\}, \{2, 2\}, \{3, 3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}$. The same manner for index b with the kl.

$$[\bar{D}^{ab}] = \begin{bmatrix} \bar{D}^{11} & \bar{D}^{12} & \bar{D}^{13} & \bar{D}^{14} & \bar{D}^{15} & \bar{D}^{16} \\ \bar{D}^{21} & \bar{D}^{22} & \bar{D}^{23} & \bar{D}^{24} & \bar{D}^{25} & \bar{D}^{26} \\ \bar{D}^{31} & \bar{D}^{32} & \bar{D}^{33} & \bar{D}^{34} & \bar{D}^{35} & \bar{D}^{36} \\ \bar{D}^{41} & \bar{D}^{42} & \bar{D}^{43} & \bar{D}^{44} & \bar{D}^{45} & \bar{D}^{46} \\ \bar{D}^{51} & \bar{D}^{52} & \bar{D}^{53} & \bar{D}^{54} & \bar{D}^{55} & \bar{D}^{56} \\ \bar{D}^{61} & \bar{D}^{62} & \bar{D}^{63} & \bar{D}^{64} & \bar{D}^{65} & \bar{D}^{66} \end{bmatrix} \quad (16)$$

rewritten the index of $\mathbb{R}^3$:

$$[\bar{D}^{ab}] = \begin{bmatrix} D^{1111} & D^{1122} & D^{1133} & D^{1112} & D^{1123} & D^{1131} \\ D^{2211} & D^{2222} & D^{2233} & D^{2212} & D^{2223} & D^{2231} \\ D^{3311} & D^{3322} & D^{3333} & D^{3312} & D^{3323} & D^{3331} \\ D^{1211} & D^{1222} & D^{1233} & D^{1212} & D^{1223} & D^{1231} \\ D^{2311} & D^{2322} & D^{2333} & D^{2312} & D^{2323} & D^{2331} \\ D^{3111} & D^{3122} & D^{3133} & D^{3112} & D^{3123} & D^{3131} \end{bmatrix} \quad (17)$$

In consequence $\bar{D}_b^a$, $\bar{D}_a^b$ y $\bar{D}_{ab}$ in matricial representation:

$$[\bar{D}_b^a] = \left[\begin{array}{ccc|ccc} D^{11}_{.,11} & D^{11}_{.,22} & D^{11}_{.,33} & 2D^{11}_{.,12} & 2D^{11}_{.,23} & 2D^{11}_{.,31} \\ D^{22}_{.,11} & D^{22}_{.,22} & D^{22}_{.,33} & 2D^{22}_{.,12} & 2D^{22}_{.,23} & 2D^{22}_{.,31} \\ D^{33}_{.,11} & D^{33}_{.,22} & D^{33}_{.,33} & 2D^{33}_{.,12} & 2D^{33}_{.,23} & 2D^{33}_{.,31} \\ D^{12}_{.,11} & D^{12}_{.,22} & D^{12}_{.,33} & 2D^{12}_{.,12} & 2D^{12}_{.,23} & 2D^{12}_{.,31} \\ D^{23}_{.,11} & D^{23}_{.,22} & D^{23}_{.,33} & 2D^{23}_{.,12} & 2D^{23}_{.,23} & 2D^{23}_{.,31} \\ D^{31}_{.,11} & D^{31}_{.,22} & D^{31}_{.,33} & 2D^{31}_{.,12} & 2D^{31}_{.,23} & 2D^{31}_{.,31} \end{array}\right] \qquad [\bar{D}_a^b] = \left[\begin{array}{ccc|ccc} D_{11}^{.,11} & D_{11}^{.,22} & D_{11}^{.,33} & D_{11}^{.,12} & D_{11}^{.,23} & D_{11}^{.,31} \\ D_{22}^{.,11} & D_{22}^{.,22} & D_{22}^{.,33} & D_{22}^{.,12} & D_{22}^{.,23} & D_{22}^{.,31} \\ D_{33}^{.,11} & D_{33}^{.,22} & D_{33}^{.,33} & D_{33}^{.,12} & D_{33}^{.,23} & D_{33}^{.,31} \\ 2D_{12}^{.,11} & 2D_{12}^{.,22} & 2D_{12}^{.,33} & 2D_{12}^{.,12} & 2D_{12}^{.,23} & 2D_{12}^{.,31} \\ 2D_{23}^{.,11} & 2D_{23}^{.,22} & 2D_{23}^{.,33} & 2D_{23}^{.,12} & 2D_{23}^{.,23} & 2D_{23}^{.,31} \\ 2D_{31}^{.,11} & 2D_{31}^{.,22} & 2D_{31}^{.,33} & 2D_{31}^{.,12} & 2D_{31}^{.,23} & 2D_{31}^{.,31} \end{array}\right]$$

$$[\bar{D}_{ab}] = \left[\begin{array}{ccc|ccc} D_{1111} & D_{1122} & D_{1133} & 2D_{1112} & 2D_{1123} & 2D_{1131} \\ D_{2211} & D_{2222} & D_{2233} & 2D_{2212} & 2D_{2223} & 2D_{2231} \\ D_{3311} & D_{3322} & D_{3333} & 2D_{3312} & 2D_{3323} & 2D_{3331} \\ 2D_{1211} & 2D_{1222} & 2D_{1233} & 4D_{1212} & 4D_{1223} & 4D_{1231} \\ 2D_{2311} & 2D_{2322} & 2D_{2333} & 4D_{2312} & 4D_{2323} & 4D_{2331} \\ 2D_{3111} & 2D_{3122} & 2D_{3133} & 4D_{3112} & 4D_{3123} & 4D_{3131} \end{array}\right] \tag{18}$$

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The stress and strain is contravariats y covariants components, for example:

$$\bar{s}^a = 2G\bar{I}^{dev,ab} : \bar{e}_b \tag{19}$$

For the elastic tensor in $\mathbb{R}^6$

$$\begin{aligned}
\bar{D}^{ab} &= 3K\bar{I}^{vol,ab} + 2G\bar{I}^{dev,ab} \\
\bar{D}_{ab} &= 3K\bar{I}^{vol}_{ab} + 2G\bar{I}^{dev}_{ab} \\
\bar{D}^{a}_{.b} = \bar{D}^{.b}_{a} &= 3K\bar{I}^{vol,a}_{.b} + 2G\bar{I}^{dev}_{.b}
\end{aligned} \tag{20}$$

Invariant principle in numerical integration

- Example 1:

$$\lambda = f(\sigma' : \varepsilon')$$

$$\lambda = f([\bar{\sigma}'^{a}]^{T} \cdot [\bar{\varepsilon}'^{b})]$$

- Example 2

$$\dot{\varepsilon}^{p} = \lambda \frac{\partial f}{\partial \sigma} \quad \text{will be} \quad \bar{\dot{\varepsilon}}^{p}_{b} = \lambda \bar{I}_{ba} : \left(\frac{\bar{\partial f}}{\partial \sigma} \right)^{a}$$

- Example 3:

$$\dot{\sigma} = D : (\dot{\varepsilon} - \dot{\varepsilon}^{p})$$

will be:

$$\bar{\dot{\sigma}}^{a} = \bar{D}^{ab} : (\bar{\dot{\varepsilon}}_{b} - \bar{\dot{\varepsilon}}^{p}_{b}) \quad \text{or} \quad \bar{\dot{\sigma}}^{a} = \bar{D}^{a}_{.b} : [\bar{I}^{bc} : (\bar{\dot{\varepsilon}}_{c} - \bar{\dot{\varepsilon}}^{p}_{c})]$$

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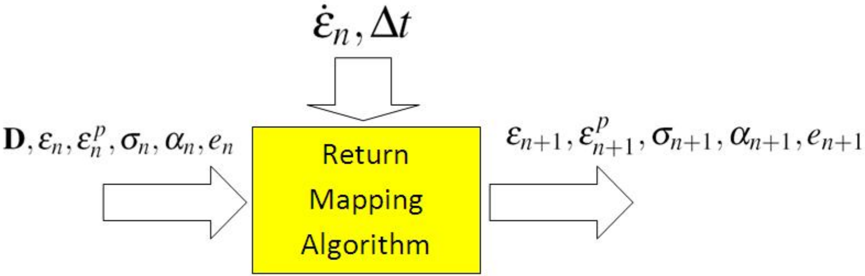
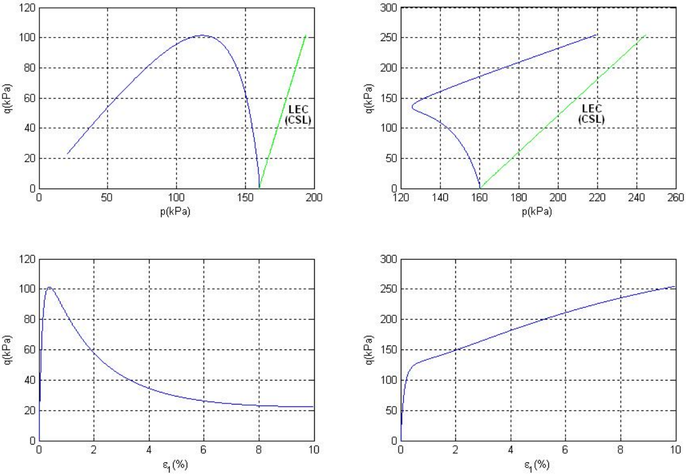


$\dot{\varepsilon}_n, \Delta t$
$\mathbf{D}, \varepsilon_n, \varepsilon_n^p, \sigma_n, \alpha_n, e_n$
Return Mapping Algorithm
$\varepsilon_{n+1}, \varepsilon_{n+1}^p, \sigma_{n+1}, \alpha_{n+1}, e_{n+1}$

Elastic	G_0	3,14x10^4
	K_0	3,14x10^4
	a	0,6
Critical state	M_c	1,14
	λ	0,025
	$(e_c)_{ref}$	0,80
State Parameter	k_c^b	3,975
	k_c^d	4,20
Hardening	h_0	1200
	m	0,05
Dilatancy	A	0,54



q(kPa)
p(kPa)
LEC
(CSL)
q(kPa)
p(kPa)
LEC
(CSL)
q(kPa)
ε_1(%)
q(kPa)
ε_1(%)

Conclude Remarks

- The results show that for the analyzed particular case, there is a saving on time until 15 % in $\mathbb{R}^6$ in relation to $\mathbb{R}^3$. This is due to the smaller access to memory keeping and recovering data, as well as the diminution in the number of operations conducted by the transformation of $\mathbb{R}^3$ to $\mathbb{R}^6$.
- The transformations done of the representation of tensors in $\mathbb{R}^6$ of covariants to contravariants and inversely, they can be easily made indicating the terms that are altered instead of making tensors multiplications with the identity tensors (covariants, contravariants or mixed), since these last multiplications are expensive in terms of computacional time.

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