Cosserat Continuum

The Cosserat continuum and the shear bands formation in granular material under cyclic loading

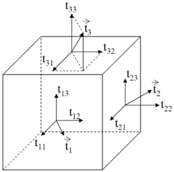
Alfonso Mariano Ramos Cañón[1]

[1]Departamento de Ingeniería Civil y Ambiental
Universidad de Los Andes

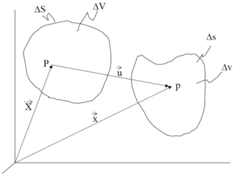
December 6th de 2007

Contents

Motivation

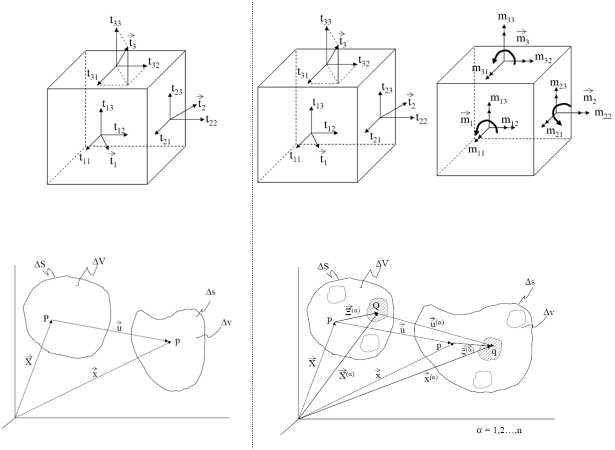


t_{33}
$\vec{t}_3$
t_{32}
t_{31}
t_{23}
$\vec{t}_2$
t_{13}
t_{22}
t_{12}
t_{21}
t_{11}
$\vec{t}_1$



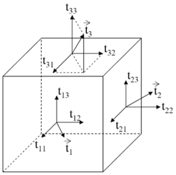
ΔS
ΔV
Δs
Δv
P
$\vec{u}$
p
$\vec{X}$
$\vec{x}$

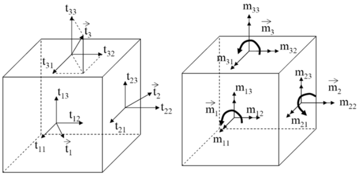
Motivation



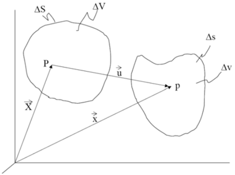
t33
t3
t32
t31
t23
t2
t22
t13
t12
t21
t11
t1
m33
m3
m32
m31
m23
m2
m22
m13
m1
m12
m21
m11
ΔS
ΔV
Δs
Δv
P
p
u
X
x
Q
q
Ξ(α)
u(α)
ξ(α)
X(α)
x(α)
α = 1,2…,n

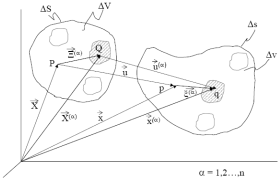
Motivation



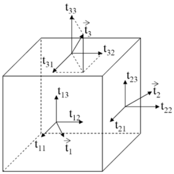


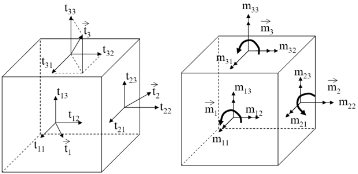
Constitutive equation





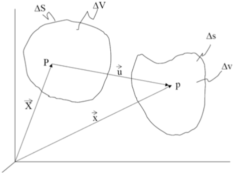
Motivation

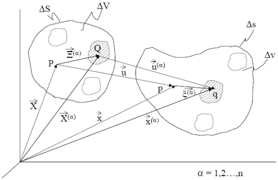




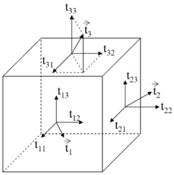
Constitutive equation

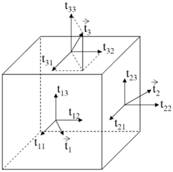
Micropolar elasticity

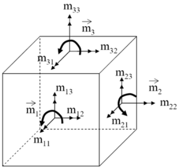




Motivation

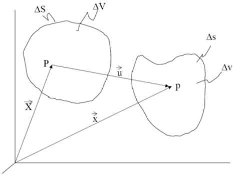


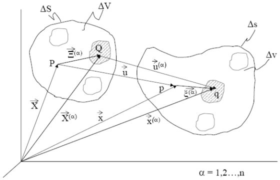




Constitutive equation

Micropolar elasticity ⇨ Hypoplasticity





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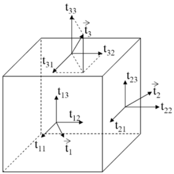
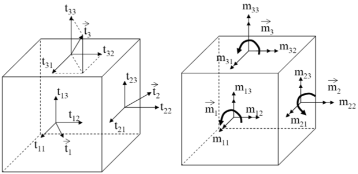
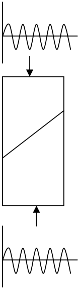


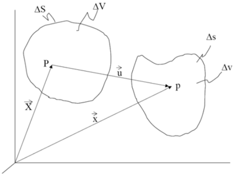


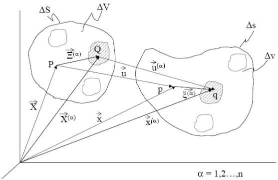
Constitutive equation

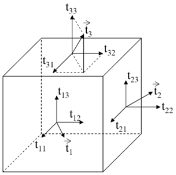
Micropolar elasticity

Hypoplasticity









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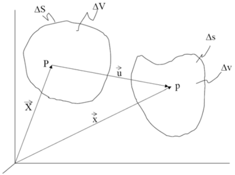
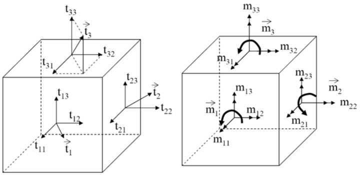
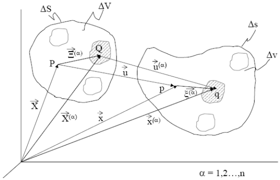
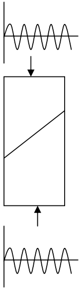




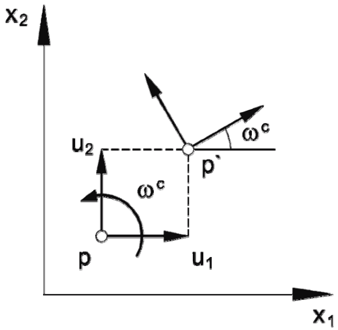
Micropolar elasticity ⇨ Hypoplasticity ⇨

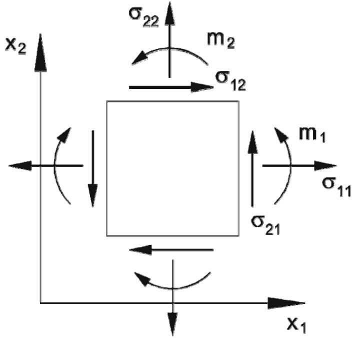


Experimentation



Motivation





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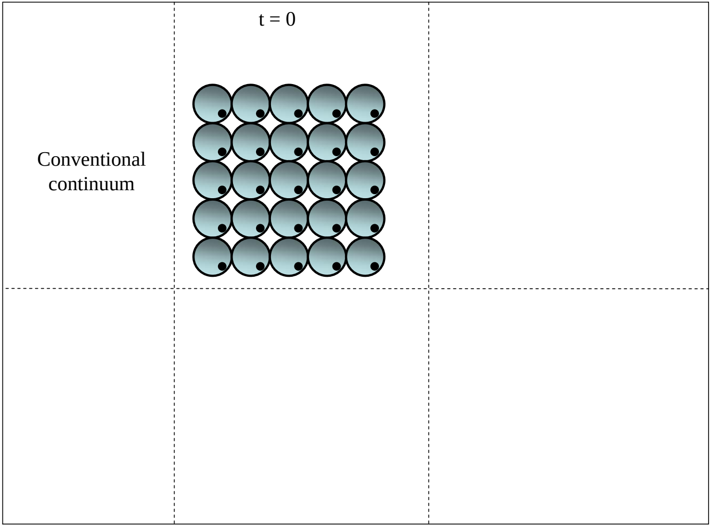
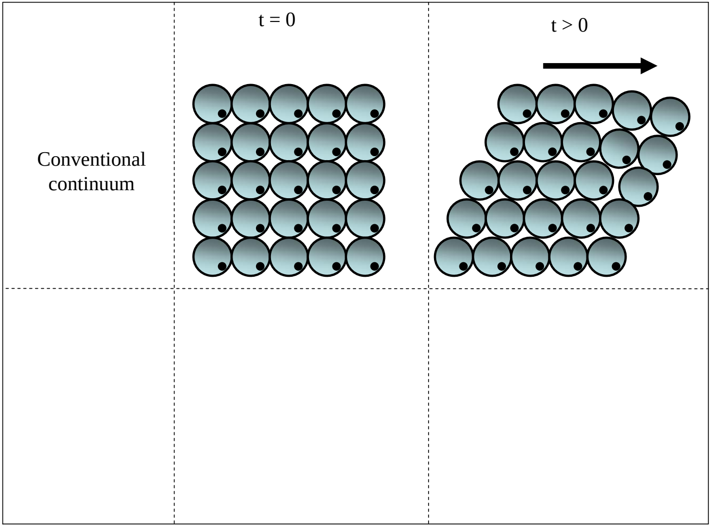
Motivation



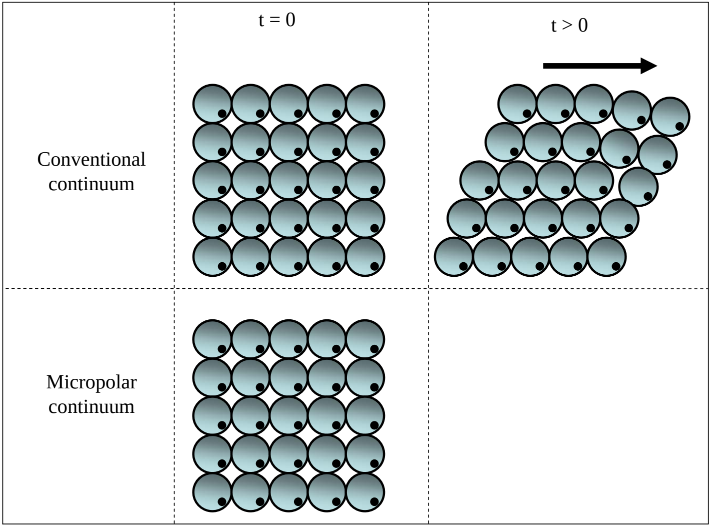
Conventional and Polar Continuum

Motivation



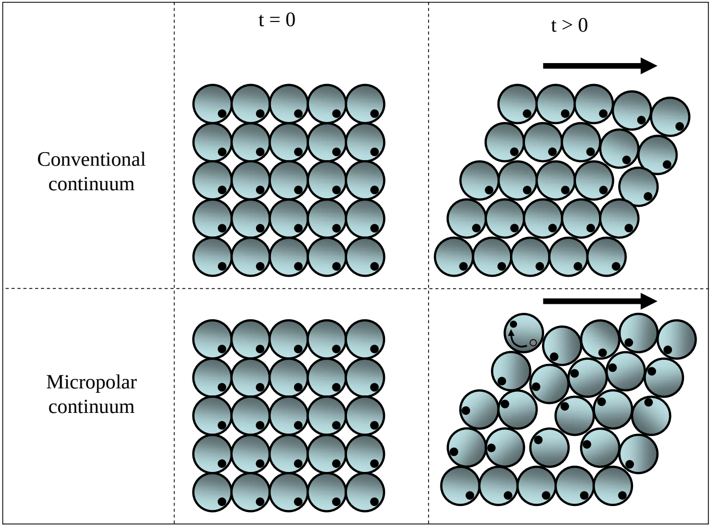
Conventional and Polar Continuum

Motivation



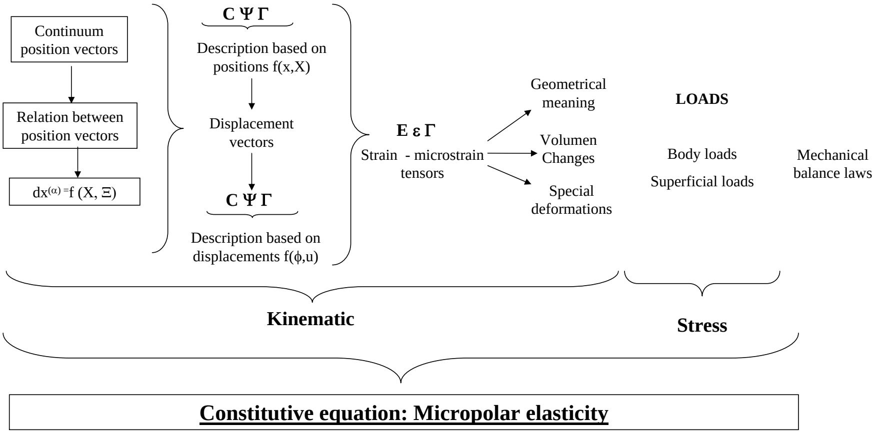
Conventional and Polar Continuum

Motivation

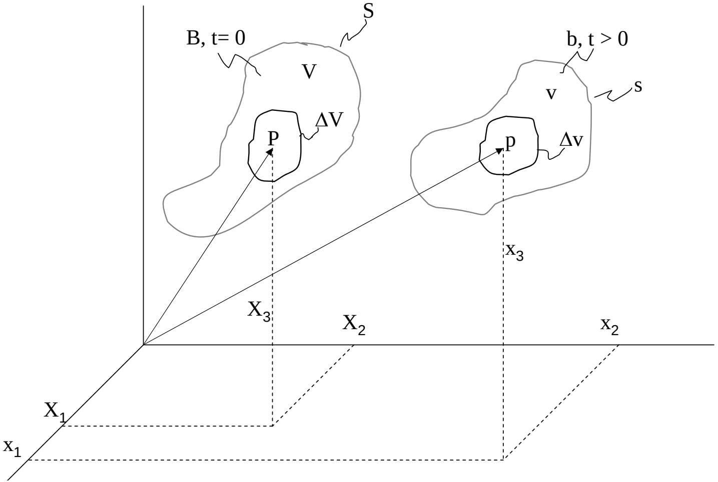


Conventional and Polar Continuum

Contents

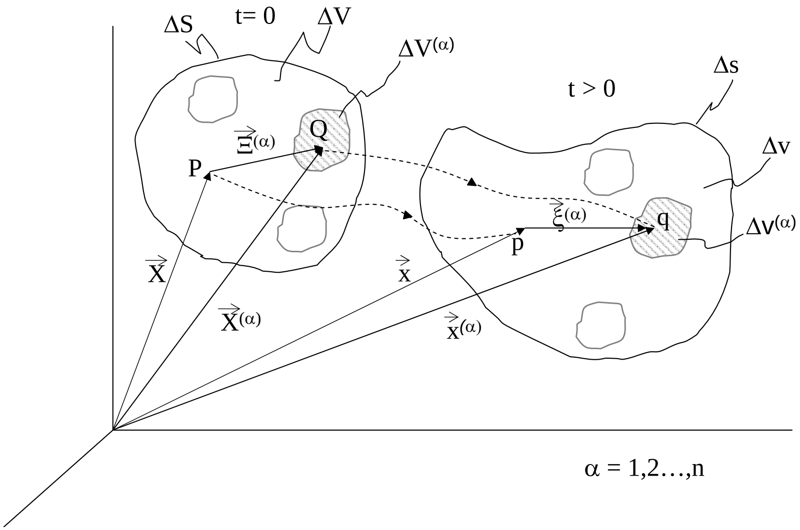

Continuum position vectors
Relation between position vectors
dx(α) =f (X, Ξ)
C Ψ Γ
Description based on positions f(x,X)
Displacement vectors
C Ψ Γ
Description based on displacements f(ϕ,u)
E ε Γ
Strain - microstrain tensors
Geometrical meaning
Volumen Changes
Special deformations
LOADS
Body loads
Superficial loads
Mechanical balance laws
Kinematic
Stress
Constitutive equation: Micropolar elasticity

Deformation and Microdeformation - Material and spatial coordinates



Material and spatial coordinates

Deformation and Microdeformation



Deformation of a microvolume

$$\overrightarrow{X}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi}^{(\alpha)} \tag{1}$$

Where $\overrightarrow{\Xi}^{(\alpha)}$ is the position of a point in the microelement relative to the center of mass of ΔV.
The final position of the αth particle will be:

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{x} + \overrightarrow{\xi}^{(\alpha)} \tag{2}$$

The motion of the center of mass P of ΔV is:

$$\overrightarrow{x} = \overrightarrow{x}(\overrightarrow{X}, t) \tag{3}$$

The relative position vector $\overrightarrow{\xi}^{(\alpha)}$ depends not only on $\overrightarrow{X}$ y t but also on $\overrightarrow{\Xi}^{(\alpha)}$, i.e.,

$$\overrightarrow{\xi}^{(\alpha)} = \overrightarrow{\xi}^{(\alpha)}(\overrightarrow{X}, \overrightarrow{\Xi}^{(\alpha)}, t) \tag{4}$$

The materials points in ΔV undergo a deformation about the centre of mass.

$$\overrightarrow{\xi}^{(\alpha)} = \underline{\chi}(\overrightarrow{X}, t)\,\overrightarrow{\Xi}^{(\alpha)} \tag{5}$$

In coordinate form, for the spatial position of a material point in a microelement, we have:

$$x_k^{(\alpha)} = x_k(\overrightarrow{X}, t) + \chi_{kK}(\overrightarrow{X}, t)\Xi_K^{(\alpha)} \tag{6}$$

$$k, K = 1, 2, 3$$

We introduce the inverse micromotions Λ_{Kk} such that

$$\chi_{kK}\Lambda_{Kl} = \delta_{kl}, \quad \chi_{kK}\Lambda_{Lk} = \delta_{KL}, \quad \chi_{kK}\Lambda_{Lk} = \underline{1} \tag{7}$$

In component form Eq. 5 reads

$$\xi_k^{(\alpha)} = \chi_{kK}(\overrightarrow{X}, t)\Xi_K^{(\alpha)} \tag{8}$$

Using Eq. 7 in the Eq. 8, we get

$$\Xi_K^{(\alpha)} = \Lambda_{Kk}(\overrightarrow{x}, t)\xi_k^{(\alpha)} \tag{9}$$

The motion and the inverse motion of a material point in a microelement are therefore expressed by

$$x_k^{(\alpha)} = x_k(\overrightarrow{X}, t) + \chi_{kK}(\overrightarrow{X}, t)\Xi_K^{(\alpha)} \tag{10}$$

$$X_K^{(\alpha)} = X_K(\overrightarrow{x}, t) + \Lambda_{Kk}(\overrightarrow{x}, t)\xi_k^{(\alpha)} \tag{11}$$

The differential line element in the deformed body is calculated.

$$x_k^{(\alpha)} = x_k(\overrightarrow{X}, t) + \chi_{kL}(\overrightarrow{X}, t)\Xi_L^{(\alpha)}$$

$$dx_k^{(\alpha)} = \left(\frac{\partial x_k}{\partial X_K} + \Xi_L \frac{\partial \chi_{kL}}{\partial X_K} \right) dX_K + \chi_{kL} d\Xi_L \qquad (12)$$

The square of the arc length is calculated by

$$(ds^{(\alpha)})^2 = d\overrightarrow{x}^{(\alpha)} \cdot d\overrightarrow{x}^{(\alpha)} = dx_k^{(\alpha)} \cdot dx_k^{(\alpha)}$$

$$\begin{aligned}(ds^{(\alpha)})^2 \quad = \quad & \left[\left(\frac{\partial x_k}{\partial X_K} + \Xi_M \frac{\partial \chi_{kM}}{\partial X_K}\right) dX_K + \chi_{kK} d\Xi_K\right] \cdot \\ & \left[\left(\frac{\partial x_k}{\partial X_L} + \Xi_N \frac{\partial \chi_{kN}}{\partial X_L}\right) dX_L + \chi_{kL} d\Xi_L\right]\end{aligned}$$

Strain and Microstrain Tensors

$$(ds^{(\alpha)})^2 = \left[C_{KL} + 2\Gamma_{KML}\Xi_M + \Xi_M \frac{\partial \chi_{kM}}{\partial X_K}\Xi_N \frac{\partial \chi_{kN}}{\partial X_L}\right] dX_K dX_L + 2\left[\Psi_{KL} + \chi_{kL}\frac{\partial \chi_{kM}}{\partial X_K}\right] dX_K d\Xi_L + \chi_{kL}\chi_{kK} d\Xi_L d\Xi_k$$

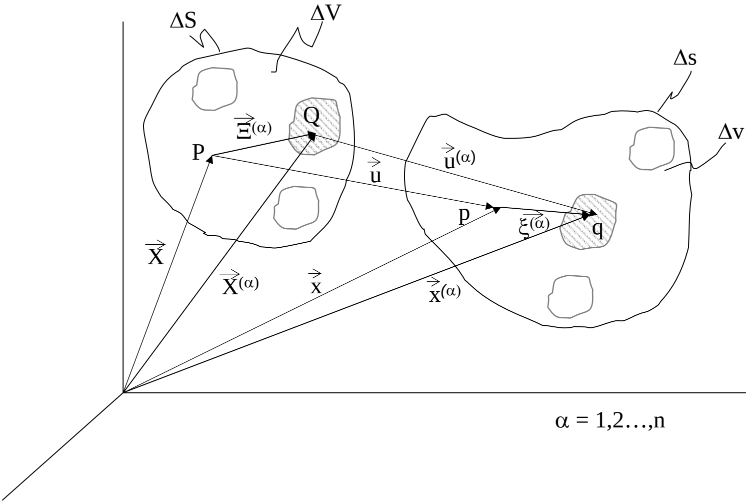
Where:

$$C_{KL}(\overrightarrow{X}, t) \equiv \frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L} \quad (13)$$

$$\Psi_{KL}(\overrightarrow{X}, t) \equiv \frac{\partial x_k}{\partial X_K}\chi_{kL} \quad (14)$$

$$\Gamma_{KLM}(\overrightarrow{X}, t) \equiv \frac{\partial x_k}{\partial X_K}\frac{\partial \chi_{kM}}{\partial X_L} \quad (15)$$

Strain and Microstrain Tensors



Displacement vectors

Strain and Microstrain Tensors

From the last figure:

$$\overrightarrow{X} + \overrightarrow{U} = \overrightarrow{x} \Rightarrow \quad \overrightarrow{U} = \overrightarrow{x} - \overrightarrow{X} \tag{16}$$

$$\Xi^{(\alpha)} + \overrightarrow{U}^{(\alpha)} = \overrightarrow{U} + \overrightarrow{\xi}^{(\alpha)}$$

$$\overrightarrow{U}^{(\alpha)} = \overrightarrow{U} + \overrightarrow{\xi}^{(\alpha)} - \overrightarrow{\Xi}^{(\alpha)} \tag{17}$$

$$x_K = U_K + X_K \tag{18}$$

From Eq 18 , by partial differentiation, we obtain

$$\frac{\partial x_k}{\partial X_K} = \frac{\partial U_k}{\partial X_K} + \frac{\partial X_k}{\partial X_K} = \frac{\partial U_k}{\partial X_K} + \delta_{kK} \quad (19)$$

Similarly, we introduce the microdisplacement tensors $\Phi_{LK}(\overrightarrow{X},t)$

$$\chi_{kK} = \delta_{kK} + \Phi_{kK} \quad (20)$$

By use of Eqs. 16, 20 the Eq. 17 may also expressed as

$$U_K^{(\alpha)} = U_K + \Xi_K \Phi_{kK} \quad (21)$$

Strain and Microstrain Tensors

$$C_{KL} \approx \frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L} + \delta_{KL} \tag{22}$$

$$\Psi_{KL} \approx \frac{\partial U_L}{\partial X_K} + \delta_{KL} + \Phi_{KL} \tag{23}$$

$$\Gamma_{KLM} \approx \frac{\partial \Phi_{KM}}{\partial X_L} \tag{24}$$

The material strain tensor E_{KL} and the material microstrain tensors $\mathscr{E}_{KL}$ and Γ_{KLM} are defined thus:

$$E_{KL} \equiv \frac{1}{2}(C_{KL} - \delta_{KL}) = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L} + \delta_{KL} - \delta_{KL}\right) \quad (25)$$

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L}\right) \quad (26)$$

$$\mathscr{E}_{KL} \equiv \Psi_{KL} - \delta_{KL} = \frac{\partial U_L}{\partial X_K} + \delta_{KL} + \Phi_{KL} - \delta_{KL} = \frac{\partial U_L}{\partial X_K} + \Phi_{KL} \quad (27)$$

$$\Gamma_{KLM} \equiv \frac{\partial \Phi_{KM}}{\partial X_L} \quad (28)$$

The tensor Φ_{KL} is defined as antisymmetric for the theory of micropolar elasticity.

$$\Phi_{KL} = -\Phi_{LK} \tag{29}$$

in the spatial notation, $\phi_{kl} = -\phi_{lk}$. Every skew-symmetric, second tensor Φ_{KL} can be expressed by an axial vector Φ_K defined by:

$$\Phi_K = \frac{1}{2} e_{KLM} \Phi_{ML} \tag{30}$$

Eq 30 can be expressed as:

$$\Phi_1 = \Phi_{32}, \quad \Phi_2 = \Phi_{13}, \quad \Phi_3 = \Phi_{21}$$

The Eq 30 can be written as

$$\Phi_{KL} = -e_{KLM} \Phi_M \tag{31}$$

Substituting into Eq 20, we have

$$\chi_{kK} = \delta_{kK} + \Phi_{kK} = \delta_{kK} - e_{kKM}\Phi_M \tag{32}$$

In the classical theory, we have the rotational tensor

$$R_{KL} = -R_{LK} \equiv \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} - \frac{\partial U_L}{\partial X_K}\right) \tag{33}$$

The axial vector R_K

$$R_K = \frac{1}{2}e_{KLM}R_{ML} \quad R_{KL} = -e_{KLM}R_M \tag{34}$$

Using Eqs. 25 y 34, we obtain

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L}\right) \quad R_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} - \frac{\partial U_L}{\partial X_K}\right)$$

$$\frac{\partial U_K}{\partial X_L} = E_{KL} + R_{KL}$$

$$\frac{\partial U_K}{\partial X_L} = E_{KL} - e_{KLM} R_M \tag{35}$$

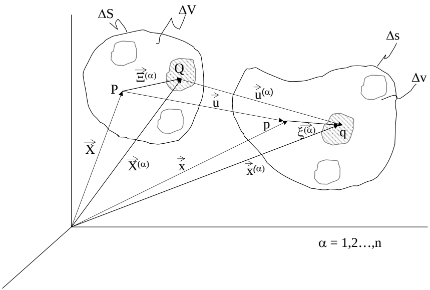
Micropolar Strains and Rotations

When this and Eq. 31 are substituted into Eq. 27 and 28 we get

$$\mathscr{E}_{KL} = E_{KL} + e_{KLM}(R_M - \Phi_M) \tag{36}$$

$$\Gamma_{KLM} = \frac{\partial \Phi_{KL}}{\partial X_M} = \frac{\partial(-e_{KLN}\Phi_N)}{\partial X_M} = -e_{KLN}\frac{\partial \Phi_N}{\partial X_M} \tag{37}$$

Regarding this figure (Displacement vectors) and the Eq. 21:

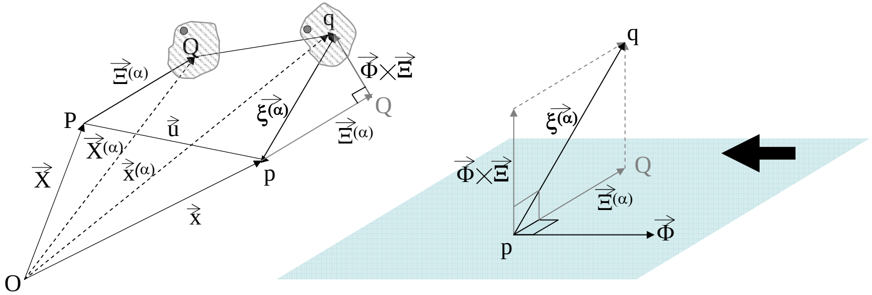


Displacement Vectors

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi} + \overrightarrow{U}^{(\alpha)} \quad U_K^{(\alpha)} = U_K + \Phi_{kK}\Xi_K$$

$$\underline{\Phi}\,\overrightarrow{\Xi} = \overrightarrow{\Phi} \times \overrightarrow{\Xi}$$

$\underline{\Phi}$ is antisymmetric.

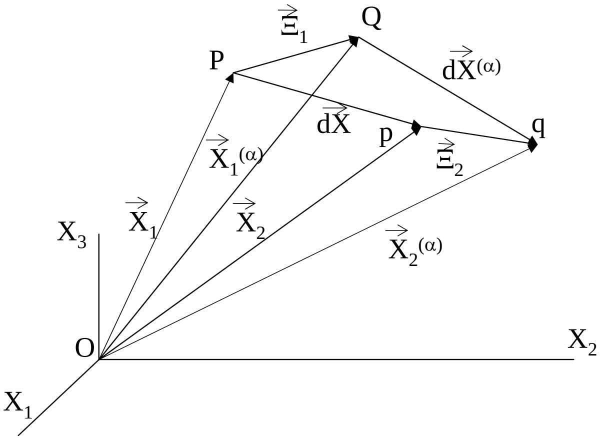


Microrotations

the spatial position of the αth point $\overrightarrow{x}^{(\alpha)}$ is

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{U} + \underbrace{\overrightarrow{\Xi} + \overrightarrow{\Phi} \times \overrightarrow{\Xi}}_{\overrightarrow{\xi}}$$

Consider the deformation of an infinitesimal vector $\overrightarrow{dX}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi}$ at $\overrightarrow{X} + \overrightarrow{\Xi}$.



Deformation of an infinitesimal vector

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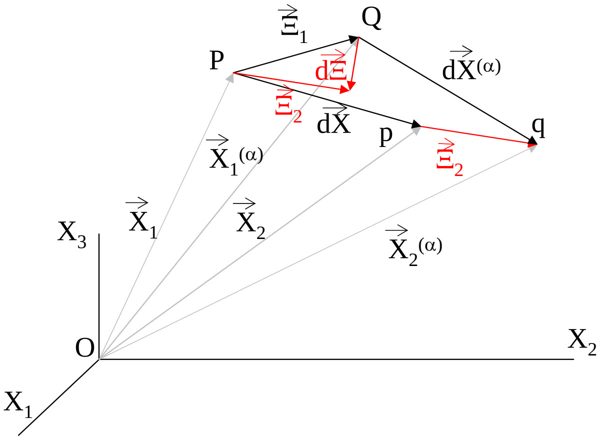

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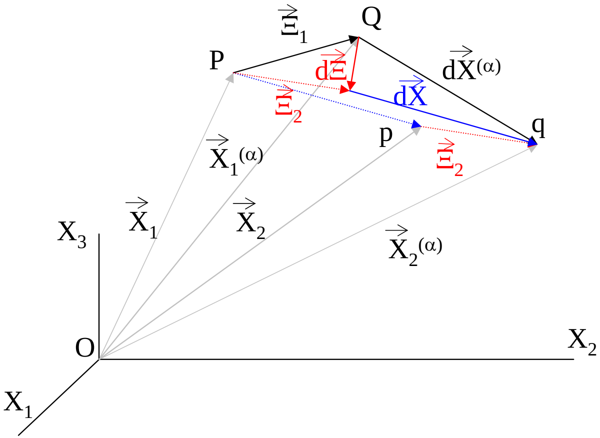


Deformation of an infinitesimal vector

Micropolar Strains and Rotations

On deformation, the vector $\overrightarrow{dX}^{(\alpha)}$ becomes:

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} + \frac{\partial \overrightarrow{U}}{\partial X_K} dX_K - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} - \overrightarrow{\Xi} \times \frac{\partial \overrightarrow{\Phi}}{\partial X_K} dX_K \qquad (38)$$

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} + \Gamma_{(KM)}) dX_K \qquad (39)$$

Now, it is defined a new minirotation vector $\vec{\Gamma}$ by

$$\Gamma_K \equiv \frac{1}{2} e_{KLM} \Gamma_{ML}, \quad \Gamma_{[KL]} = -e_{KLM} \Gamma_M \tag{40}$$

This vector is called minirotation for distinction from the microrotation Φ. Eq.40 can be written:

$$\Gamma_K = \frac{1}{2} \left(-\frac{\partial \Phi_L}{\partial X_L} \Xi_K + \frac{\partial \Phi_K}{\partial X_L} \Xi_L \right) \tag{41}$$

Micropolar Strains and Rotations

Macrostrain tensor (Eq. 25)

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L}\right)$$

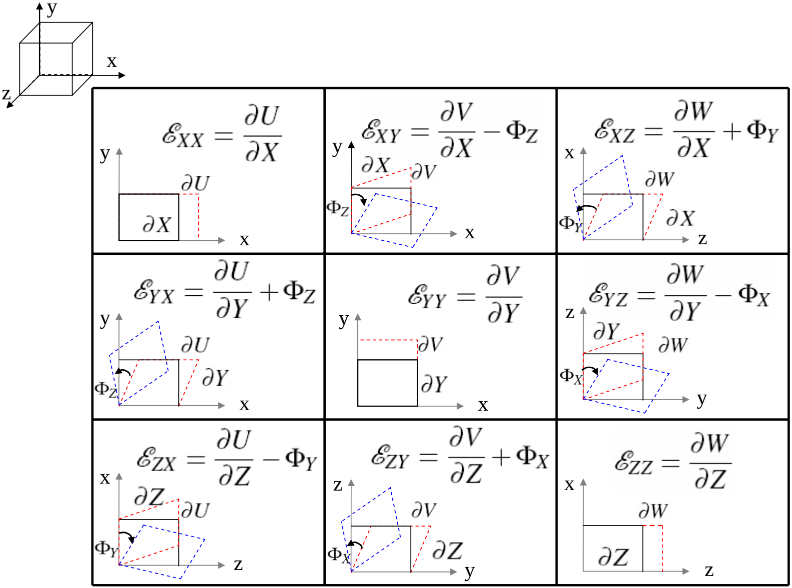
$$\begin{aligned}
E_{XX} &= \frac{\partial U}{\partial X} & E_{XY} &= \frac{1}{2}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) \\
E_{YY} &= \frac{\partial V}{\partial Y} & E_{YZ} &= \frac{1}{2}\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right) \\
E_{ZZ} &= \frac{\partial W}{\partial Z} & E_{ZX} &= \frac{1}{2}\left(\frac{\partial W}{\partial X} + \frac{\partial U}{\partial Z}\right)
\end{aligned}$$

Micropolar Strains and Rotations

Micropolar strain tensor (Eq. 27)

$$\mathscr{E}_{KL} = \frac{\partial U_L}{\partial X_K} + \Phi_{KL} \quad \Phi_{KL} = -e_{KLM}\Phi_M \rightarrow \mathscr{E}_{KL} = \frac{\partial U_L}{\partial X_K} - e_{KLM}\Phi_M$$

$\mathscr{E}_{XX} = \frac{\partial U}{\partial X}$	$\mathscr{E}_{XY} = \frac{\partial V}{\partial X} - \Phi_Z$	$\mathscr{E}_{XZ} = \frac{\partial W}{\partial X} + \Phi_Y$
$\mathscr{E}_{YX} = \frac{\partial U}{\partial Y} + \Phi_Z$	$\mathscr{E}_{YY} = \frac{\partial V}{\partial Y}$	$\mathscr{E}_{YZ} = \frac{\partial W}{\partial Y} - \Phi_X$
$\mathscr{E}_{ZX} = \frac{\partial U}{\partial Z} - \Phi_Y$	$\mathscr{E}_{ZY} = \frac{\partial V}{\partial Z} + \Phi_X$	$\mathscr{E}_{ZZ} = \frac{\partial W}{\partial Z}$



Microdeformation Tensor

Micropolar Strains and Rotations

Micropolar strain tensor of third order (Eq. 28)

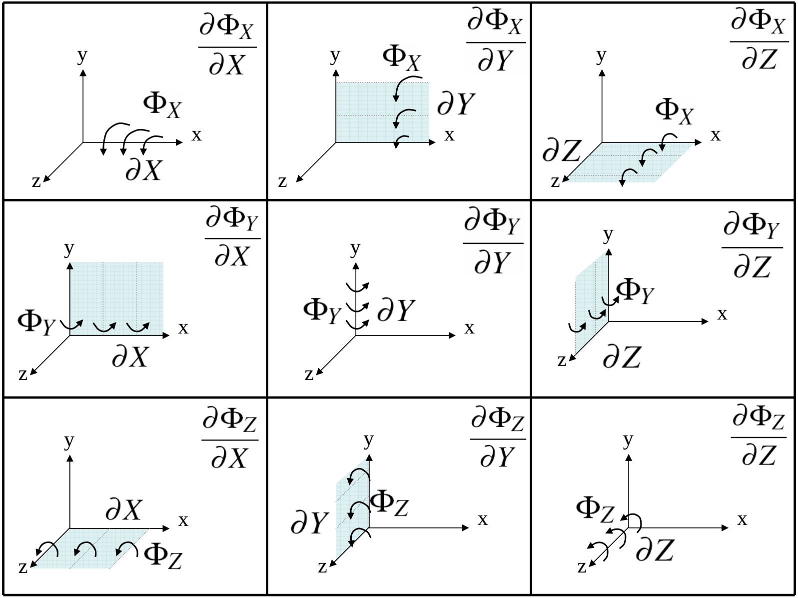
$$\Gamma_{KLM} = \frac{\partial \Phi_{KL}}{\partial X_M} \quad \Phi_{KL} = -e_{KLN}\Phi_N \rightarrow \Gamma_{KLM} = \frac{\partial(-e_{KLN}\Phi_N)}{\partial X_M} \quad (42)$$

$$\Gamma_{YZX} = -\Gamma_{ZYX} = \frac{\partial \Phi_X}{\partial X}; \Gamma_{YZY} = -\Gamma_{ZYY} = \frac{\partial \Phi_X}{\partial Y}; \Gamma_{YZZ} = -\Gamma_{ZYZ} = \frac{\partial \Phi_X}{\partial Z}$$

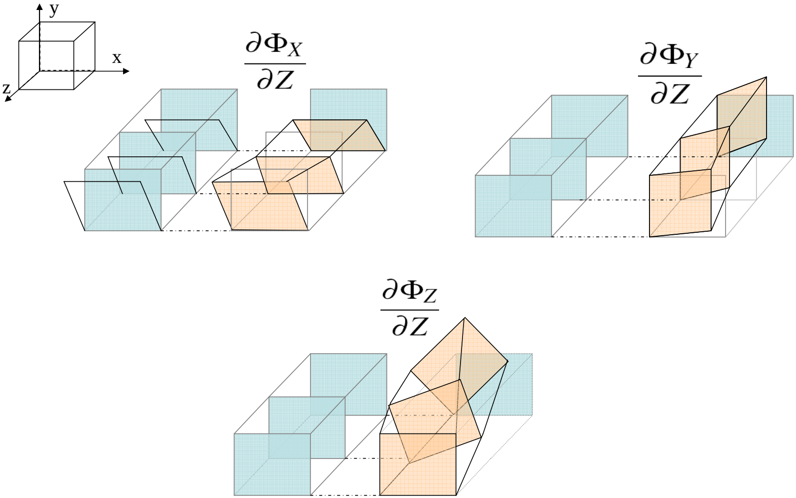
$$\Gamma_{ZXX} = -\Gamma_{XZX} = \frac{\partial \Phi_Y}{\partial X}; \Gamma_{ZXY} = -\Gamma_{XZY} = \frac{\partial \Phi_Y}{\partial Y}; \Gamma_{ZXZ} = -\Gamma_{XZZ} = \frac{\partial \Phi_Y}{\partial Z}$$

$$\Gamma_{XYX} = -\Gamma_{YXX} = \frac{\partial \Phi_Z}{\partial X}; \Gamma_{XYY} = -\Gamma_{YXY} = \frac{\partial \Phi_Z}{\partial Y}; \Gamma_{XYZ} = -\Gamma_{YXZ} = \frac{\partial \Phi_Z}{\partial Z}$$

all others $\Gamma_{KLM} = 0$



Micropolar strain tensor of third order


y
x
z
$\frac{\partial \Phi_X}{\partial Z}$
$\frac{\partial \Phi_Y}{\partial Z}$
$\frac{\partial \Phi_Z}{\partial Z}$

Macrorotation vector (Eq. 34)

$$R_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} - \frac{\partial U_L}{\partial X_K}\right)$$

$$R_X = \frac{1}{2}\left(\frac{\partial W}{\partial Y} - \frac{\partial V}{\partial Z}\right); R_Y = \frac{1}{2}\left(\frac{\partial U}{\partial Z} - \frac{\partial W}{\partial X}\right); \quad R_Z = \frac{1}{2}\left(\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y}\right)$$

Microrotation vector

$$\overrightarrow{\Phi} = \Phi_X \hat{I} + \Phi_Y \hat{J} + \Phi_Z \hat{K}$$

Micropolar Strains and Rotations

Minirotation vector (Eq. 41)

$$\Gamma_X = \frac{1}{2}\left[-\left(\frac{\partial \Phi_Y}{\partial Y} + \frac{\partial \Phi_Z}{\partial Z}\right)\Xi_X + \frac{\partial \Phi_X}{\partial Y}\Xi_Y + \frac{\partial \Phi_X}{\partial Z}\Xi_Z\right]$$

$$\Gamma_Y = \frac{1}{2}\left[-\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Z}{\partial Z}\right)\Xi_Y + \frac{\partial \Phi_Y}{\partial X}\Xi_X + \frac{\partial \Phi_Y}{\partial Z}\Xi_Z\right]$$

$$\Gamma_Z = \frac{1}{2}\left[-\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Y}{\partial Y}\right)\Xi_Z + \frac{\partial \Phi_Z}{\partial X}\Xi_X + \frac{\partial \Phi_Z}{\partial Y}\Xi_Y\right]$$

Other way to see the minirotation vector $\overrightarrow{\Gamma}$ is the tensorial form: $\overrightarrow{\Gamma} = \frac{1}{2}\underline{B}\,\overrightarrow{\Xi}$:

$$\overrightarrow{\Gamma} = \begin{bmatrix} \Gamma_X \\ \Gamma_Y \\ \Gamma_Z \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} -\left(\frac{\partial \Phi_Y}{\partial Y} + \frac{\partial \Phi_Z}{\partial Z}\right) & \frac{\partial \Phi_X}{\partial Y} & \frac{\partial \Phi_X}{\partial Z} \\ \frac{\partial \Phi_Y}{\partial X} & -\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Z}{\partial Z}\right) & \frac{\partial \Phi_Y}{\partial Z} \\ \frac{\partial \Phi_Z}{\partial X} & \frac{\partial \Phi_Z}{\partial Y} & -\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Y}{\partial Y}\right) \end{bmatrix}$$

$$\overrightarrow{\Xi} = \begin{bmatrix} \Xi_X \\ \Xi_Y \\ \Xi_Z \end{bmatrix}$$

Geometrical meaning of Micropolar strains

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} + \Gamma_{(KM)}) dX_K$$

The deformation of the vector $\overrightarrow{dX}^{(\alpha)} \equiv \overrightarrow{dX} + \overrightarrow{d\Xi}$ may be achieved by the following three operations:

1. A rigid translation of $\overrightarrow{dX} + \overrightarrow{d\Xi}$ from material centroid $\overrightarrow{X}$ to the spatial centroid $\overrightarrow{x}$.
2. Rigid rotations of $\overrightarrow{dX}$ and $\overrightarrow{d\Xi}$ by the amounts $d\overrightarrow{X} \times (\overrightarrow{R} + \overrightarrow{\Gamma})$ and $\overrightarrow{d\Xi} \times \overrightarrow{\Phi}$, respectively.
3. Finally, a stretch represented by the strains E_{KL} and $\Gamma_{(KL)}$

Geometrical meaning of Micropolar strains

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} + \Gamma_{(KM)})dX_K$$

Eq. 39 can be written as

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dx} + \overrightarrow{dy} + \overrightarrow{dz}$$

Where

$$\overrightarrow{dx} = \overrightarrow{dX} - \overrightarrow{dX} \times \overrightarrow{R} + E_{KL} dX_K \qquad (43)$$

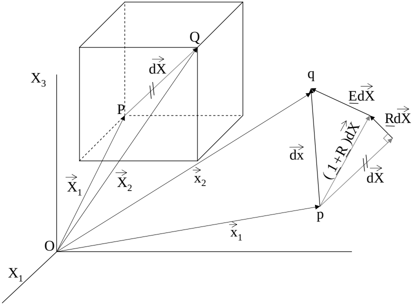
$$\overrightarrow{dy} = -\overrightarrow{dX} \times \overrightarrow{\Gamma} + \Gamma_{(KM)} dX_K \qquad (44)$$

$$\overrightarrow{dz} = d\,\overrightarrow{\Xi} - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} \qquad (45)$$

Geometrical meaning of Micropolar strains and Rotations

Eq. 43 can be expressed as

$$\overrightarrow{dx} = \overrightarrow{dX} + \overrightarrow{R} \times \overrightarrow{dX} + E_{KL} dX_K = \overrightarrow{dX} + \underline{R}\overrightarrow{dX} + \underline{E}\overrightarrow{dX} = \overrightarrow{dX}\left[\underline{1} + \underline{R}\right] + \underline{E}\overrightarrow{dX}$$

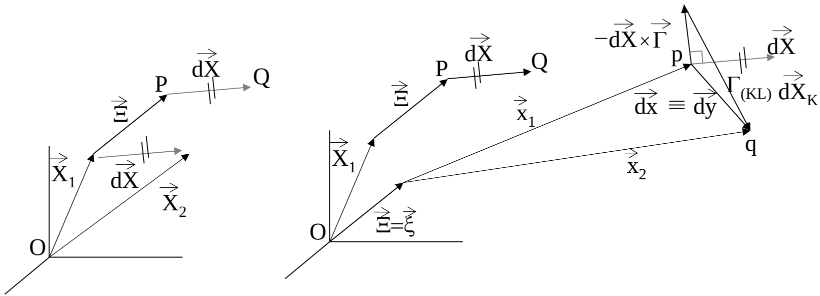


Conventional Continnum

Geometrical meaning of Micropolar strains and Rotations

Eq. 44 can be written as

$$\overrightarrow{dy} = -\overrightarrow{dX} \times \overrightarrow{\Gamma} + \Gamma_{(KL)}\overrightarrow{dX} = \Gamma_{[KL]}\overrightarrow{dX} + \Gamma_{(KL)}\overrightarrow{dX}$$

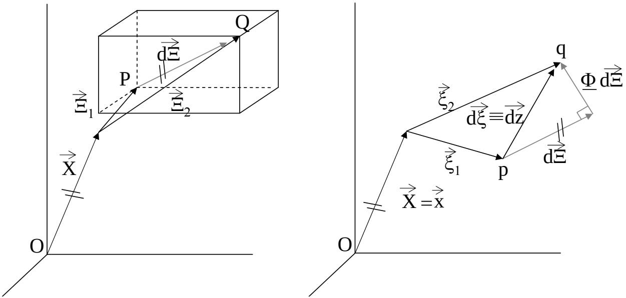


Ξ is constant

Geometrical meaning of Micropolar strains and Rotations

Eq. 45 can be expressed as

$$\overrightarrow{dz} = d\,\overrightarrow{\Xi} - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} = d\,\overrightarrow{\Xi} + \overrightarrow{\Phi} \times \overrightarrow{d\Xi} = \underbrace{d\,\overrightarrow{\Xi} + \underline{\Phi}\overrightarrow{d\Xi}}_{d\,\overrightarrow{\Xi}\,(\underline{1}+\underline{\Phi})}$$



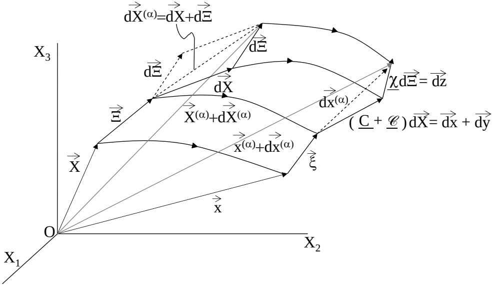
X is a constant

Geometrical meaning of Micropolar strains and Rotations

$$dx_k^{(\alpha)} = \left(\frac{\partial x_k}{\partial X_K} + \Xi_L \frac{\partial \chi_{kL}}{\partial X_K} \right) dX_K + \chi_{kL} d\Xi_L$$

Eq. 12 can be written as

$$\overrightarrow{dx}^{(\alpha)} = (\underline{C} + \underline{\mathscr{C}}) \overrightarrow{dX} + \underline{\chi} \overrightarrow{d\Xi} \quad \mathscr{C} = \frac{\partial \chi_{kL}}{\partial X_K} \Xi_L$$



Macrodeformation and microdeformation

Volumen Changes

Material macrovolume element with Ξ fixed

$$dV_o \equiv dX_1 dX_2 dX_3$$

Material minivolume element with X fixed

$$d\mathscr{V}_o \equiv d\Xi_1 d\Xi_2 d\Xi_3$$

After deformation $dV_o \to dv$ and $d\mathscr{V}_o \to d\upsilon$

$$dv = J dX_1 dX_2 dX_3 \quad d\upsilon = j d\Xi_1 d\Xi_2 d\Xi_3$$

Where j and J are the Jacobians of deformation with Ξ and X fixed, respectively

Volumen Changes

$$\overrightarrow{dx}^{(\alpha)} = (\underline{C} + \underline{\mathscr{C}})\overrightarrow{dX} + \underline{\chi}\overrightarrow{d\Xi} \quad \mathscr{C} = \frac{\partial \chi_{kL}}{\partial X_K}\Xi_L$$

$$J \equiv \det\left(\frac{\partial x_k}{\partial X_K} + \frac{\partial \chi_{kM}}{\partial X_K}\Xi_M\right) \tag{46}$$

$$j \equiv \det(\chi_{kM}) \tag{47}$$

$$J \simeq 1 + E_{KK} + \Gamma_{KK} \tag{48}$$

$$J \simeq 1 + E_{KK} + \Gamma_{KK} = 1 + tr\underline{E} + (\nabla \times \overrightarrow{\Phi}) \cdot \overrightarrow{\Xi}$$

$$\frac{dv}{dVo} - 1 = tr\underline{E} + (\nabla \times \overrightarrow{\Phi}) \cdot \overrightarrow{\Xi} \tag{49}$$

Taking Eq. 47 and calculating the determinant

$$j = \{\det[\chi_{kK}\chi_{kL}]\}^{1/2} \tag{50}$$

Using the Eq. 20 $\chi_{kK} = \delta_{kK} + \Phi_{kK}$

$$j \approx \{\det[\delta_{KL} + \Phi_{KL} + \Phi_{LK}]\}^{1/2} = 1 + \Phi_{KK} \tag{51}$$

$\Phi_{KK} = 0$, so that

$$\frac{d\upsilon}{d\mathscr{V}_o} - 1 = 0 \tag{52}$$

$\underline{E} = \underline{\mathscr{E}} = \underline{0}, \quad \underline{\Gamma} = \underline{0}$. If $E_{KL} = 0$ implies U is constant, then R_{KL} not depends on X.

If $\Gamma_{KL} = 0 \rightarrow \Gamma_{KLM} = 0 \quad \Gamma_{KM} = \Gamma_{KLM} \underbrace{\Xi_L}_{\neq 0}$

If $\Gamma_{KLM} = 0 \rightarrow \Phi_N$ is constant because $\underbrace{\Gamma_{KLM}}_{=0} = -e_{KLN} \underbrace{\frac{\partial \overbrace{\Phi_N}^{Constant}}{\partial X_M}}_{=0}$

If $\underline{\mathscr{E}} = 0$, from Eq. 36 we have:

$\underbrace{\mathscr{E}_{KL}}_{=0} = \underbrace{E_{KL}}_{=0} + e_{KLM}(R_M - \Phi_M) \rightarrow R_M = \Phi_M \rightarrow R_K = \Phi_K$ where

$R_K = \frac{1}{2} e_{KLM} R_{ML}$; R_{ML} and R_K are independents of X

Macroisochoric deformation: if the material macrovolumen remains unchanged.
Miniisochoric deformation: if the material minivolumen remains unchanged. j is always equal to one.
From Eq. 48:

$$J \simeq 1 + E_{KK} + \Gamma_{KK}$$

$$E_{KK} + \Gamma_{KK} = 0 \tag{53}$$

The Eq. 53 is valid, if E_{KK} and Γ_{KK} are equal to zero.

$$E_{KK} = \nabla \cdot U = 0$$

From Eq. 49 is obtained that $\Gamma_{KK} = (\nabla \times \overrightarrow{\Phi}) \cdot \overrightarrow{\Xi}$. Then $(\nabla \times \overrightarrow{\Phi}) = 0$

Some special deformations - Homogeneous strain

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{x} + \overrightarrow{\xi} \quad x_k = D_{kK} X_K \quad \xi_k = \mathfrak{D}_{kK} \Xi_K$$

$$\overrightarrow{x}^{(\alpha)} = \underbrace{D_{kK}}_{constant} X_K + \underbrace{\mathfrak{D}_{kK}}_{constant} \Xi_K \quad = \underline{D}\,\overrightarrow{X} + \underline{\mathfrak{D}}\,\overrightarrow{\Xi} \tag{54}$$

$x_k = D_{kK} X_K$ is the homogeneous strain in classical continuum
$\xi_k = \mathfrak{D}_{kK} \Xi_K$ is the microhomogeneous deformation.

Homogeneous strain - Simple Shear

$$\underline{D} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -\infty < S < \infty \tag{55}$$

$$\overrightarrow{x}^{(\alpha)} = \underline{D}\overrightarrow{X} + \underline{\mathfrak{D}}\,\overrightarrow{\Xi}$$

$$\begin{bmatrix} x_1^{(\alpha)} \\ x_2^{(\alpha)} \\ x_3^{(\alpha)} \end{bmatrix} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 & -\mathfrak{D}_3 & \mathfrak{D}_2 \\ \mathfrak{D}_3 & 1 & -\mathfrak{D}_1 \\ -\mathfrak{D}_2 & \mathfrak{D}_1 & 1 \end{bmatrix} \begin{bmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{bmatrix}$$

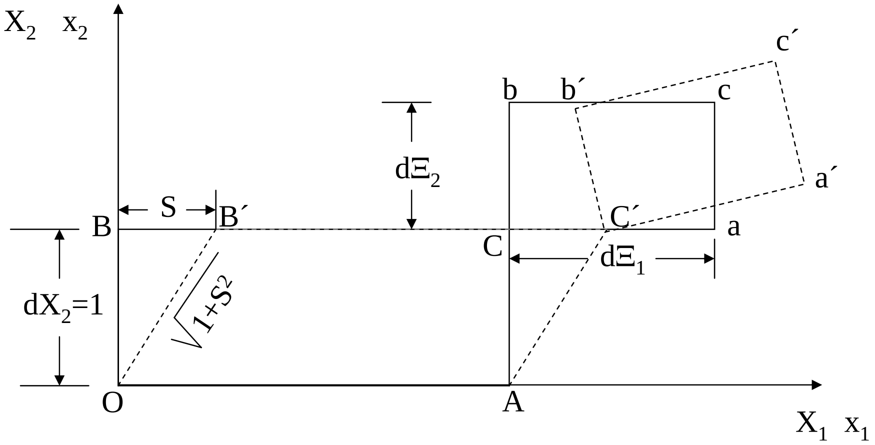
$$\begin{aligned} x_1^{\alpha} &= X_1 + SX_2 + \Xi_1 + \mathfrak{D}_2\Xi_3 - \mathfrak{D}_3\Xi_2 \\ x_2^{\alpha} &= X_2 + \Xi_2 + \mathfrak{D}_3\Xi_1 - \mathfrak{D}_1\Xi_3 \\ x_3^{\alpha} &= X_3 + \Xi_3 + \mathfrak{D}_1\Xi_2 - \mathfrak{D}_2\Xi_1 \end{aligned} \tag{56}$$

Homogeneous strain - Simple Shear

$$C_{KL} = D_{kK} D_{kL}$$

$$\underline{C} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{C} = \begin{bmatrix} 1 & S & 0 \\ S & 1+S^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (57)$$

Homogeneous strain - Simple Shear



Simple Shear

Velocity of a material point

Definition: The material derivative of any tensor is defined as the partial derivative of that tensor with respect to time with the material coordinates X_k and Ξ_K held constant.

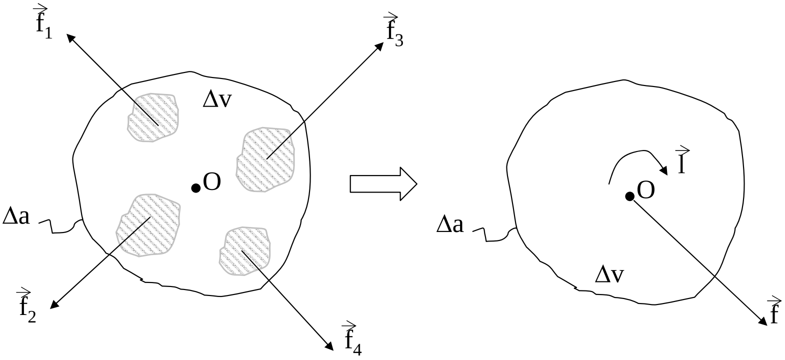
$$\vec{\xi} = \underline{\chi}\,\vec{\Xi} \tag{58}$$

$$\vec{\dot{\xi}} = \underline{\dot{\chi}}(X,t)\,\vec{\Xi} \tag{59}$$

$$\vec{\dot{\xi}} = -\vec{\xi} \times \underbrace{\vec{\nu}}_{\dot{\phi}} \tag{60}$$

Loads

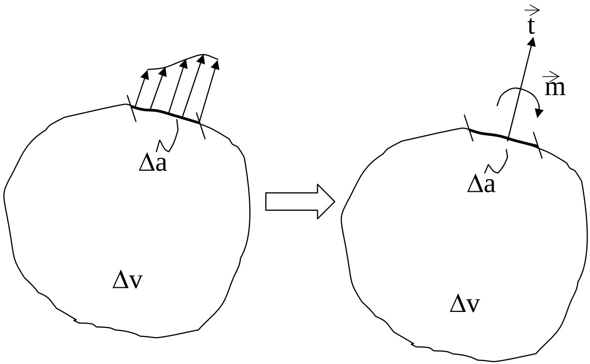
Body loads



Macrovolume element with force and couple body

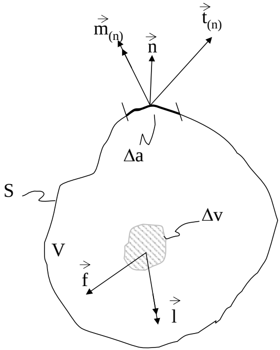
$\vec{f}$ = Body force per unit mass; $\vec{l}$ = Body couple per unit mass

Surface loads



Macrovolume element with surface loads

Surface and body loads



Surface and body loads on volumen

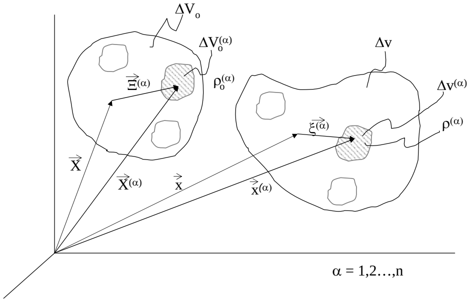
$$\mathfrak{F} = \oint_s \overrightarrow{t}_{(n)} da + \int_v \rho \overrightarrow{f} dv$$

$$\mathfrak{M} = \oint_s [\overrightarrow{m}_{(n)} + \overrightarrow{x} \times \overrightarrow{t}_{(n)}] da + \int_v \rho (\overrightarrow{l} + \overrightarrow{x} \times \overrightarrow{f}) dv$$

Mechanical balance laws

1. Conservation of mass
2. Balance of momentum
3. Balance of momentum of momentum

The total mass of each microelement remains constant during any deformation.



Macromass elements containing micromass elements

$$\rho_o^{(\alpha)} \Delta V_o^{(\alpha)} = \rho^{(\alpha)} \Delta v^{(\alpha)} \tag{61}$$

The total mass of a macrovolume before and after deformation, is given by

$$\rho_o \Delta V_o \equiv \sum_{\alpha=1}^{N} \rho_o^{(\alpha)} \Delta V_o^{(\alpha)} \tag{62}$$

$$\rho \Delta v \equiv \sum_{\alpha=1}^{N} \rho^{(\alpha)} \Delta v^{(\alpha)} \tag{63}$$

$$\rho_o \Delta V_o = \rho \Delta v \tag{64}$$

If ΔV_o and $\Delta v \rightarrow dV_o$ and dv then

$$\rho_o dV_o = \rho dv \quad \frac{\rho_o}{\rho} = \frac{dv}{dV_o} \equiv J \tag{65}$$

$$\int_V \rho_o dV_o = \int_v \rho dv \tag{66}$$

The time rate of change of momentum is equal to the sum of all forces acting on a body

$$\Delta p = \sum_{\alpha} \rho^{(\alpha)} \mathbf{v}^{(\alpha)} \Delta v^{(\alpha)} = \sum_{\alpha} \rho^{\alpha} (\mathbf{v} + \dot{\xi}) \Delta v^{(\alpha)}$$

$$dp = \rho \mathbf{v} dv$$

The total momentum of the body is

$$p = \int_v \rho \mathbf{v} dv \tag{67}$$

The principle of balance of momentum is expressed by

$$\frac{d}{dt} \int_v \rho \mathbf{v} dv = \oint_s t_{(n)} da + \int_v \rho f dv \tag{68}$$

Principle of balance of moment of momentum

The time rate of change of moment of momentum about a point is equal to the sum of all couples and the moment of all forces about that point.

The mechanical moment of momentum of a microelement is

$$\overrightarrow{x}^{(\alpha)} \times \rho^{(\alpha)}\mathbf{v}^{(\alpha)}\Delta v^{(\alpha)}$$

The total moment of momentum of a macroelement is calculated by:

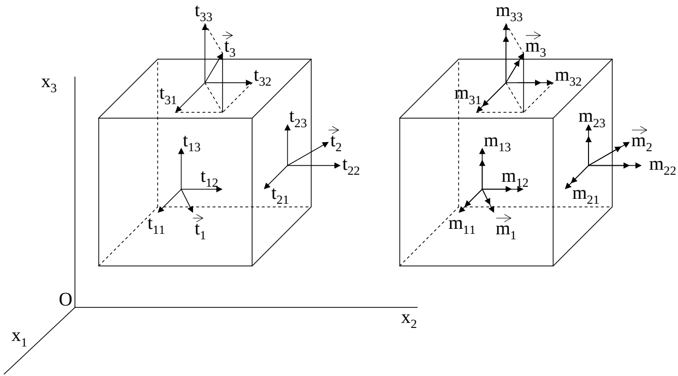
$$\Delta\mathfrak{M} = \sum_{\alpha} \overrightarrow{x}^{(\alpha)} \times \rho^{(\alpha)}\mathbf{v}^{(\alpha)}\Delta v^{(\alpha)} \quad (69)$$

$$\Delta\mathfrak{M} = \sum_{\alpha} (\overrightarrow{x} + \overrightarrow{\xi}) \times \rho^{(\alpha)}(\mathbf{v} + \overrightarrow{\dot{\xi}})\Delta v^{(\alpha)} \quad (70)$$

The principle of moment of momentum is expressed by

$$\frac{d}{dt}\int_{v}(\overrightarrow{x} \times \rho\mathbf{v} + \rho\sigma)dv = \oint_{s}(\overrightarrow{x} \times t_{(n)} + m_{(n)})da + \int_{v}\rho(l + \overrightarrow{x} \times \overrightarrow{f})dv \quad (71)$$

Stress and couple stress



Stress and couple stress

Stress and couple stress

$$\underline{t} = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}, \quad \underline{m} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{yx} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{bmatrix} \qquad (72)$$

Local balance laws

From the principle of balance of momentum (Eq. 68):

$$\frac{d}{dt}\int_v \rho \mathbf{v} dv = \oint_s \overrightarrow{t}_{(n)} da + \int_v \rho \overrightarrow{f} dv$$

$$\int_v \rho \overrightarrow{a} dv = \oint_s \overrightarrow{t}_{(n)} da + \int_v \rho \overrightarrow{f} dv \tag{73}$$

$$\int_v \underbrace{[Div \underline{t} + \rho(\overrightarrow{f} - \overrightarrow{a})]}_{=0} dv = 0 \tag{74}$$

$$\frac{\partial t_{lk}}{\partial X_l} + \rho(f_k - a_k) = 0 \tag{75}$$

Local balance laws

From principle of moment of momentum (Eq. 71)

$$\frac{d}{dt}\int_v(\overrightarrow{x}\times\rho\,\overrightarrow{\mathbf{v}}+\rho\,\overrightarrow{\sigma})dv=\oint_s(\overrightarrow{x}\times\overrightarrow{t_{(n)}}+\overrightarrow{m_{(n)}})da+\int_v\rho(\overrightarrow{l}+\overrightarrow{x}\times\overrightarrow{f})dv \tag{76}$$

$$\frac{\partial m_{lk}}{\partial x_l}+\rho(l_k-\dot{\sigma}_k)+e_{ilk}t_{lk}=0 \tag{77}$$

$$\frac{\partial t_{lk}}{\partial x_l}+\rho(f_k-a_k)=\overrightarrow{0}$$

Local balance laws

$$\begin{aligned}
\frac{\partial t_{xx}}{\partial x}+\frac{\partial t_{yx}}{\partial y}+\frac{\partial t_{zx}}{\partial z}+\rho(f_x-a_x) &= 0\\
\frac{\partial t_{xy}}{\partial x}+\frac{\partial t_{yy}}{\partial y}+\frac{\partial t_{zy}}{\partial z}+\rho(f_y-a_y) &= 0\\
\frac{\partial t_{xz}}{\partial x}+\frac{\partial t_{yz}}{\partial y}+\frac{\partial t_{zz}}{\partial z}+\rho(f_z-a_z) &= 0
\end{aligned} \tag{78}$$

$$\begin{aligned}
\frac{\partial m_{xx}}{\partial x}+\frac{\partial m_{yx}}{\partial y}+\frac{\partial m_{zx}}{\partial z}+t_{yz}-t_{zy}+\rho(l_x-\dot{\sigma}_x) &= 0\\
\frac{\partial m_{xy}}{\partial x}+\frac{\partial m_{yy}}{\partial y}+\frac{\partial m_{zy}}{\partial z}+t_{zx}-t_{xz}+\rho(l_y-\dot{\sigma}_y) &= 0\\
\frac{\partial m_{xz}}{\partial x}+\frac{\partial m_{yz}}{\partial y}+\frac{\partial m_{zz}}{\partial z}+t_{xy}-t_{yx}+\rho(l_z-\dot{\sigma}_z) &= 0
\end{aligned} \tag{79}$$

In linear micropolar elasticity, the strain measures are

$$\mathscr{E}_{KL} = E_{KL} + e_{KLM}(R_M - \Phi_M) \quad (80)$$

$$\Gamma_{KLM} = e_{KLN}\frac{\partial \Phi_N}{\partial M} \quad (81)$$

Only 9 components $\frac{\partial \Phi_N}{\partial X_M}$ are independent and non zero (Eq. 42). We may, instead of Γ_{KLM}, to use the axial tensor $\frac{\partial \Phi_K}{\partial X_L}$.
For the anisotropic micropolar elastic solids, we have the following relations.

$$t_{KL} = A_{KLMN}\mathscr{E}_{MN} \quad (82)$$

$$m_{KL} = B_{KLMN}\frac{\partial \Phi_M}{\partial X_N} \quad (83)$$

If the body is isotropic, the tensors A and B have to be isotropics. The most general form of second order isotropic tensor is:

$$A_{KLMN} = A_1\delta_{KL}\delta_{MN} + A_2\delta_{KM}\delta_{LN} + A_3\delta_{KN}\delta_{LM} \tag{84}$$

$$B_{KLMN} = B_1\delta_{KL}\delta_{MN} + B_2\delta_{KM}\delta_{LN} + B_3\delta_{KN}\delta_{LM} \tag{85}$$

$$t_{KL} = A_1\delta_{KL}\mathscr{E}_{MM} + A_2\mathscr{E}_{KL} + A_3\mathscr{E}_{LK} \tag{86}$$

$$m_{KL} = B_1\delta_{KL}\frac{\partial \Phi_R}{\partial X_R} + B_2\frac{\partial \Phi_K}{\partial X_L} + B_3\frac{\partial \Phi_L}{\partial X_L} \tag{87}$$

On introducing

$$A_1 = \lambda, \quad A_2 = \mu + \kappa, \quad A_3 = \mu \tag{88}$$

$$B_1 = \alpha, \quad B_2 = \beta, \quad B_3 = \gamma \tag{89}$$

The eqs. 86 and 87 can be written as:

$$t_{KL} = \lambda \delta_{KL} \mathscr{E}_{MM} + (\mu + \kappa) \mathscr{E}_{KL} + \mu \mathscr{E}_{LK} \tag{90}$$

$$m_{KL} = \alpha \delta_{KL} \frac{\partial \Phi_R}{\partial X_R} + \beta \frac{\partial \Phi_K}{\partial X_L} + \gamma \frac{\partial \Phi_L}{\partial X_K} \tag{91}$$

Theory of micropolar elasticity

$$t_{KL} = \lambda \delta_{KL} E_{RR} + (2\mu + \kappa) E_{KL} + \kappa e_{KLM} (R_M - \Phi_M)$$

$$m_{KL} = \alpha \delta_{KL} \frac{\partial \Phi_R}{\partial X_R} + \beta \frac{\partial \Phi_K}{\partial X_L} + \gamma \frac{\partial \Phi_L}{\partial X_K}$$

There are 4 extra modulus in micropolar isotropic elasticity $\alpha, \beta, \gamma, \kappa$. If these modulus are set equal to zero, it is obtained the Hooke isotropic elasticity.

AGRADECIMIENTOS

Pontificia Universidad Javeriana
Universidad de los Andes
Colciencias - ICETEX
Prof. Arcesio Lizcano.