Implementation of the elastoplastic model Sanisand 2004 into `IncrementalDriver`

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Contents

Simple **ANI**sotropic **SAND** constitutive model

- It is one of the best elastoplastic models available
- Dilatancy (flow rule **M**) is function of η and a direct function of e !
- Fabric-dilatancy-tensor **z** increases contractancy upon unloading (anisotropic contractancy, depends on $\mathbf{z} : \dot{\boldsymbol{\epsilon}}$)
- The model is relatively well documented by cooperative authors

Further features of Sanisand

- Dependence of the peak and phase-transformation stress ratio on the state parameter $\psi = e - e_c$ similar to f_d function
- The critical void ratio e_c is dependent on the pressure p

$$e_c = e_{c0} - \lambda (p/p_{at})^{\xi} \tag{1}$$

analogously to Gudehus-Bauer formula but without a limit for $p \rightarrow \infty$

$$\frac{e_c}{e_{c0}} = \exp\left[-\left(\frac{3p}{h_s}\right)^n\right]$$

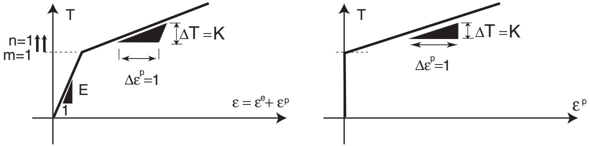
Notation for tensorial expressions

- compression is negative
- Gibbs notation (without $\otimes$)
- deviatoric part $\sqcup^*$
- derivatives $\dfrac{\partial \sqcup}{\partial \mathbf{T}} = \sqcup'$ and $\dfrac{\partial \sqcup}{\partial \boldsymbol{\alpha}} = \sqcup^{\backprime}$ ($\boldsymbol{\alpha}$ is a back stress)
- normalisation $\sqcup^{\rightarrow} = \vec{\sqcup} = \sqcup / \| \sqcup \|$
- McCauley $\langle \sqcup \rangle = \dfrac{\sqcup + |\sqcup|}{2}$

Elastoplastic framework (0)

Each elastoplastic model must define:

- elastic stress-strain relations $(\dot{\mathbf{T}} = \mathsf{E}^{\mathrm{el}} : \dot{\boldsymbol{\epsilon}})$
- yield function(s) $f(\mathbf{T}, \boldsymbol{\alpha}, \dots) = 0$
- flow rule $\mathbf{M}$ $\qquad \dot{\boldsymbol{\epsilon}}^{\mathrm{pl}} \sim \mathbf{M}$
- loading direction $\mathbf{N}$ $\qquad \mathbf{N} : \mathsf{E}^{\mathrm{el}} : \dot{\boldsymbol{\epsilon}} > 0 \rightarrow$ Loading
- hardening modulus K



1D interpretation of K. Niemunis (2008)

Elastoplastic framework (1)

All state variables $\mathbf{T}, \boldsymbol{\alpha}, e, \ldots$ at time t and the strain increment $\dot{\boldsymbol{\epsilon}}\Delta t$ are given.
All state variables at time $t + \Delta t$ are searched for, $\mathbf{T} + \dot{\mathbf{T}}\Delta t$, $\boldsymbol{\alpha} + \dot{\boldsymbol{\alpha}}\Delta t, \ldots$
The stress-strain relation is incrementally bilinear

$$\begin{cases} \overset{\circ}{\mathbf{T}} = \left[\mathsf{E}^{\mathrm{el}} - \dfrac{\mathsf{E}^{\mathrm{el}} : \mathbf{M}\ \mathbf{N} : \mathsf{E}^{\mathrm{el}}}{K + \mathbf{N} : \mathsf{E}^{\mathrm{el}} : \mathbf{M}} \right] : \dot{\boldsymbol{\epsilon}} & \text{for loading} \\ \overset{\circ}{\mathbf{T}} = \mathsf{E}^{\mathrm{el}} : \dot{\boldsymbol{\epsilon}} & \text{for un/reloading} \end{cases} \tag{2}$$

In hypoplasticity the analogous rate equation is perfectly non-linear

Elastoplastic framework (2)

The plastic strain rate $\dot{\boldsymbol{\epsilon}}^{\mathrm{pl}}$ is commonly used, although not indispensable:

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{pl}} = \mathbf{M}\dot{\lambda} = \mathbf{M}\frac{\mathbf{N} : \mathsf{E}^{\mathrm{el}} : \dot{\boldsymbol{\epsilon}}}{K + \mathbf{N} : \mathsf{E}^{\mathrm{el}} : \mathbf{M}} = \mathbf{M}\frac{\mathbf{N} : \mathsf{E}^{\mathrm{el}} : \dot{\boldsymbol{\epsilon}}}{\bar{K}} \tag{3}$$

For unloading $\dot{\boldsymbol{\epsilon}}^{\mathrm{pl}} = \mathbf{0}$. For loading $\dot{\boldsymbol{\epsilon}}^{\mathrm{pl}} \neq \mathbf{0}$ and:

$$\mathring{\mathbf{T}} = \mathsf{E}^{\mathrm{el}} : \left(\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^{\mathrm{pl}}\right) \tag{4}$$

The plastic multiplier $\dot{\lambda}$ is also frequently used, although not essential.

Elastoplastic framework (3)

Usually the hardening modulus K is defined indirectly using an evolution equation of the hardening parameters, for example

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}(\mathbf{T}, \boldsymbol{\alpha}, \ldots, \dot{\boldsymbol{\epsilon}}^{\mathrm{pl}}) \tag{5}$$

Given such hardening function, the hardening modulus is obtained as:

$$K = -\left[\frac{1}{\|f'\|} f^{\backprime} : \frac{\partial \dot{\boldsymbol{\alpha}}}{\partial \dot{\boldsymbol{\epsilon}}^{\mathrm{pl}}}\right] : \mathbf{M} \tag{6}$$

Tangential isotropic elastic stiffness is hypoelastic due to the dependance of G^{el} on the pressure:

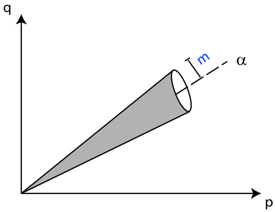
$$E^{\text{el}}_{ijkl} = G^{\text{el}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \left(K^{\text{el}} - \frac{2G^{\text{el}}}{3}\right)\delta_{ij}\delta_{kl} \qquad ☹ \quad (7)$$

Tangential elastic shear bulk moduli:

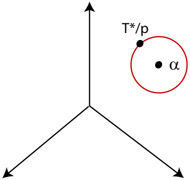
$$G^{\text{el}} = G^{\text{el}}_0(p_{\text{atm}})\frac{(2.97-e)^2}{1+e}\left(\frac{p}{p_{\text{atm}}}\right)^{\frac{1}{2}} \quad \text{and} \quad K^{\text{el}} = \frac{2(1+\nu)}{3(1-2\nu)}G^{\text{el}}, \quad (8)$$

$G^{\text{el}}_0, \nu, p_{\text{atm}} = 100$ kPa material constants

Yield Surface



Yield surface scheme

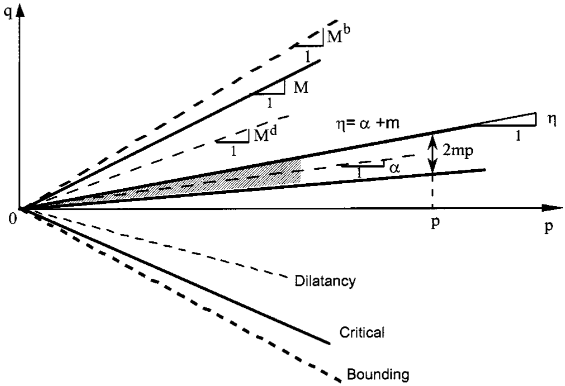


$$f \equiv \|\mathbf{T}^* - p\,\boldsymbol{\alpha}\| - mp\sqrt{\frac{2}{3}} = 0 \text{ or } f \equiv p\left[\left\|\frac{\mathbf{T}^*}{p} - \boldsymbol{\alpha}\right\| - m\sqrt{\frac{2}{3}}\right] = 0$$

with the axis in the stress space given by the tensor $\boldsymbol{\alpha}$ (back-stress tensor) $m \approx 0.05$ mat. constant

Characteristic surfaces

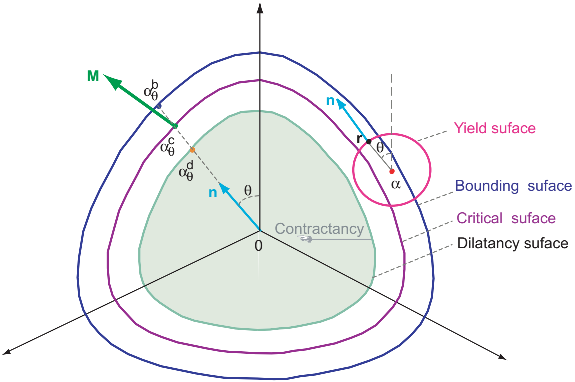
Yield, dilatancy, bounding and critical



Dafalias & Manzari, 2004

Characteristic surfaces

Three conjugated stresses $\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^c, \boldsymbol{\alpha}^d$



Dafalias & Manzari, 2004

Elastoplastic elements for Sanisand

Loading direction tensor

$$\mathbf{N} \quad = \quad (f')^{\rightarrow} \tag{9}$$

with

$$f' = \frac{\partial f}{\partial \mathbf{T}} = \frac{1}{\left\| \frac{\mathbf{T}^*}{p} - \boldsymbol{\alpha} \right\|} \left[\frac{\mathbf{T}^*}{p} - \boldsymbol{\alpha} + \frac{1}{3} \left(\frac{\mathbf{T}^*}{p} - \boldsymbol{\alpha} \right) : \frac{\mathbf{T}^*}{p} \, \mathbf{1} \right] \tag{10}$$

$$f^{\backslash} \quad = \quad \frac{\partial f}{\partial \boldsymbol{\alpha}} = -p \left(\frac{\mathbf{T}^*}{p} - \boldsymbol{\alpha} \right)^{\rightarrow} \tag{11}$$

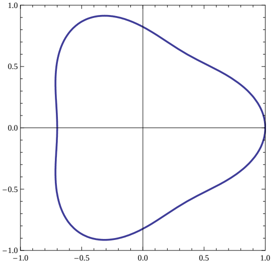
Critical state M^c

All conjugated stresses are defined as deviators proportional to $\mathbf{n} = \overrightarrow{\left(\frac{\mathbf{T}^*}{p} - \boldsymbol{\alpha}\right)}$, namely:

- The conjugated critical stress

$$\boldsymbol{\alpha}_\theta^c = \mathbf{n}\sqrt{\frac{2}{3}}\left[gM^c - m\right], \quad \text{wherein } g = g(\theta). \qquad (12)$$

$g(\theta)$ depends on the Lode angle θ of $\mathbf{n}$.



Polar plot of $g(\theta)$

Bounding and dilatancy surfaces

- Conjugated bounding stress (peak stress ratio)

$$\boldsymbol{\alpha}_\theta^b = \mathbf{n}\sqrt{\frac{2}{3}}\left(gM^c \exp(-n^b\psi) - m\right) \tag{13}$$

- Conjugated dilatancy stress (phase transition surface)

$$\boldsymbol{\alpha}_\theta^d = \mathbf{n}\sqrt{\frac{2}{3}}\left(gM^c \exp(n^d\psi) - m\right) \tag{14}$$

For $e = e_c$ we have $\psi = 0$ and $\boldsymbol{\alpha}_\theta^c = \boldsymbol{\alpha}_\theta^b = \boldsymbol{\alpha}_\theta^d$
For $\psi < 0$, $M^d < M^c < M^b$ and for $\psi > 0$, vice versa
$M^b < M^c < M^d$

Nonassociative flow rule **M**

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{pl}} \sim \mathbf{M} \tag{15}$$

The expression for the flow rule is proposed in the form

$$\mathbf{M} = \left[B\ \mathbf{n} + \frac{1}{3} D\ \mathbf{1} \right]^{\rightarrow} \tag{16}$$

The deviatoric portion **n** is scaled by:

$$B \quad = 1 + \frac{3g(\theta)}{2}\ \frac{1-c}{c} \cos 3\theta \tag{17}$$

D scales the volumetric portion, and is called the dilatancy factor

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Dilatancy function

The description of dilatancy is better than in hypoplasticity

$$D = -A_d(\boldsymbol{\alpha}_\theta^d - \boldsymbol{\alpha}) : \mathbf{n} \quad (18)$$

$$A_d = A_0(1 + \langle \mathbf{z} : \mathbf{n} \rangle) \quad (19)$$

- $\langle \mathbf{z} : \mathbf{n} \rangle$ becomes active upon reversal of loading, increasing A_d and therefore D
- The dilatancy $D > 0$ occurs for $q/p > M^d$ or for $\boldsymbol{\alpha}$ outside the dilatancy surface.

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Evolution of the backstress $\dot{\boldsymbol{\alpha}}$

$\boldsymbol{\alpha}$ evolves towards $\boldsymbol{\alpha}_\theta^b$ during loading ($\dot{\boldsymbol{\lambda}} > 0$)

$$\dot{\boldsymbol{\alpha}} = \langle\dot{\lambda}\rangle \frac{2}{3} h(\boldsymbol{\alpha}_{\boldsymbol{\theta}}^{\boldsymbol{b}} - \boldsymbol{\alpha}) = \langle\dot{\lambda}\rangle \bar{\boldsymbol{\alpha}} \quad (20)$$

The rate of evolution decreases with the distance from the latest reversal $\boldsymbol{\alpha}_{in}$

$$h = \frac{G_0^{\mathrm{el}} h_0 (1 - c_h e)}{\sqrt{p/p_{\mathrm{atm}}} (\boldsymbol{\alpha} - \boldsymbol{\alpha}_{in}) : \mathbf{n}} \quad (21)$$

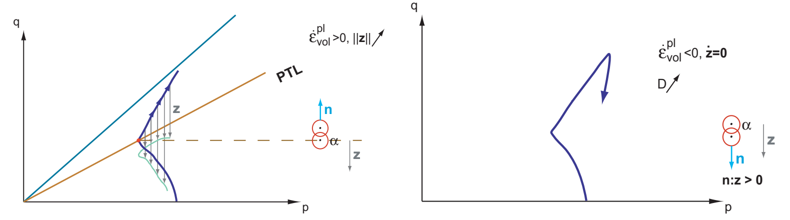
Evolution of $\mathbf{z}$

$$\dot{\mathbf{z}} = -c_z \langle \dot{\epsilon}_v^{\mathrm{pl}} \rangle (z_{\max} \mathbf{n} + \mathbf{z}) \tag{22}$$

- $\mathbf{z}$ evolves only during the dilatant plastic flow and in direction opposite to $\mathbf{n}$ up to $\mathbf{z} = -z_{\max}\mathbf{n}$
- c_z and $z_{\max}$ are mat. constants

Note that $\dot{\epsilon}_{\mathrm{vol}}^{\mathrm{pl}} \neq \mathbf{0}$ also during isochoric (undrained) tests!

Evolution of **z**



Development of **z**

Effect of **z**

Implementation

- FORTRAN explicit implementation using ABAQUS format for UMATs
- The routine was compiled together with INCREMENTALDRIVER
- Tensorial operations are easily written with the library NIEMUNISTOOLS

```
lambdad=(sv%Nbflow.xx.(EstiffEl.xx.Db))/Kbar  ! λ̇ = (1/K̄) N : E^el : ε̇
aux33  =EstiffEl  .xx.  sv%Mbflow
aux33a =  sv%Nbflow  .xx.  EstiffEl
Estiff=EstiffEl -((aux33.out.aux33a)/Kbar)  ! E^epl = E^el − (E^el : MN : E^el)/K̄
```

$$\dot{\lambda} = \frac{1}{\bar{K}} \mathbf{N} : \mathsf{E}^{\mathrm{el}} : \dot{\epsilon}$$

$$\mathsf{E}^{\mathrm{epl}} = \mathsf{E}^{\mathrm{el}} - \frac{\mathsf{E}^{\mathrm{el}} : \mathbf{MN} : \mathsf{E}^{\mathrm{el}}}{\bar{K}}$$

Useful routines for all models

```
subroutine PROPS2MAT(cmname, nprops, props, mat)   ! copy props() to mat%...+ derived constants

subroutine IMPORT(ntens, nstatv, statev, stress, dstran, dtime,
mat, Db, Tb, sv)  ! dstran,stress,statev → D, T, sv

subroutine EXPORT(mat, ctrl, ntens, nstatv, Db, LLout, Tb, sv, dtime,
ddsdde, statev, stress)       ! dstran,stress,statev → D, T, sv
```

Material parameters - Toyoura sand

15 parameters are required in Sanisand:

1.	G_0^{el}	Elastic shear modulus	125 kPa
2.	ν	Poisson coefficient	0.05
3.	M_c	Critical back-stress ratio in triaxial compression	1.25
4.	c	Ratio of extension to compression quantities	0.712
5.	e_{c0}	Void ratio at $p = 0$ on the CSL	0.934
6.	λ	Critical state line material constant	0.019
7.	ξ	Critical state line material constant	0.70

Material parameters - Toyoura sand

8.	n^d	material constant needed to calculate the stress image on the dilatancy surface	3.50
9.	A_0	Dilatancy material constant	0.704
10.	n^b	material constant needed to calculate the stress image on the boundary surface	1.1
11.	h_0	Positive material constant to define the hardening modulus	7.05
12.	c_h	Positive material constant to define the hardening modulus	0.968
13.	m	Tangent of half the opening angle of the yield surface	0.05

Material parameters - Toyoura sand

14.	z_{max}	Maximum value that **z** can attain	4
15.	c_z	Control of the pace evolution of **z**	600

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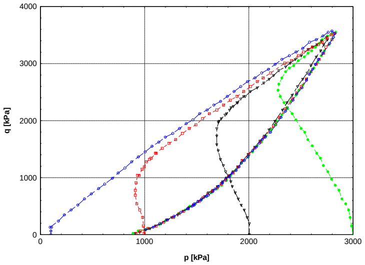
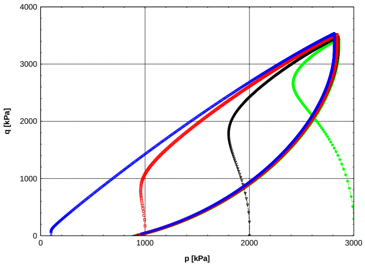
Monotonic undrained triaxial tests

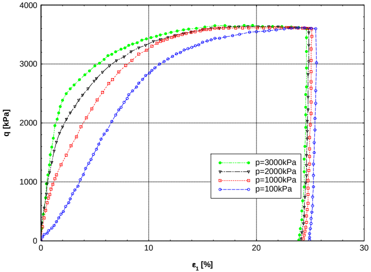


Verdugo (1996) $e_0 = 0.735, D_r = 0.64$



Sanisand 2004

Monotonic undrained triaxial tests



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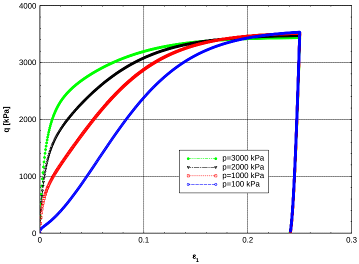
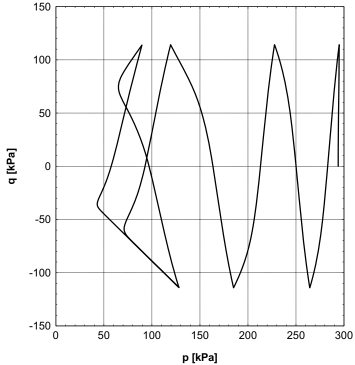


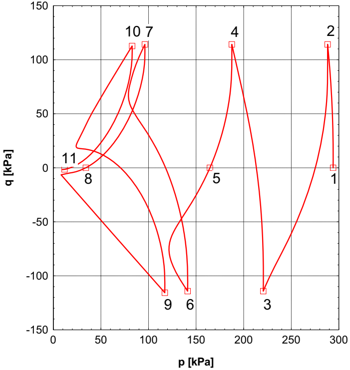
Sanisand 2004

Butterfly attractor

Undrained cyclic triaxial tests



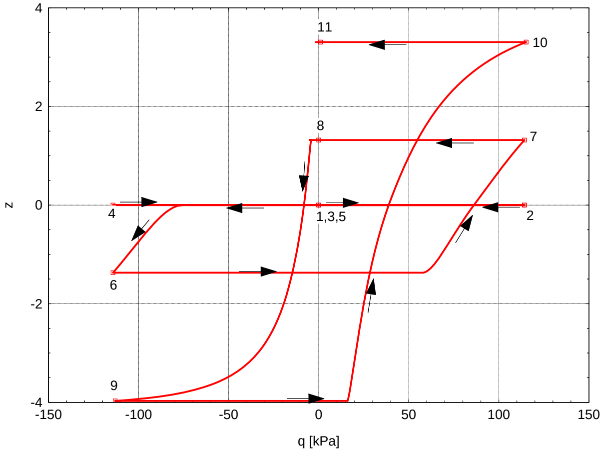
Hypoplasticity



Sanisand 2004

Evolution of **z**

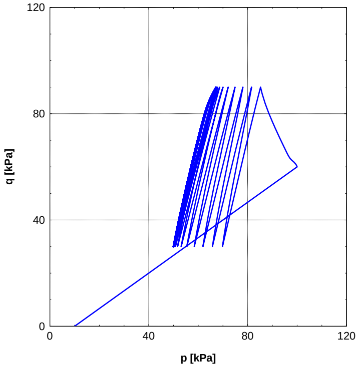
Undrained cyclic triaxial tests



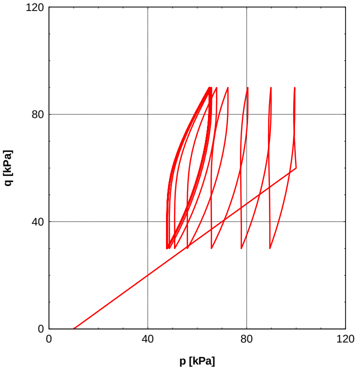
z evolution, undrained cyclic triaxial

Lentil attractor

Unsymmetric undrained cyclic triaxial tests

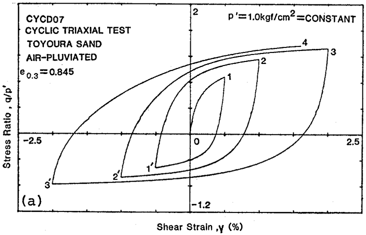


Hypoplasticity

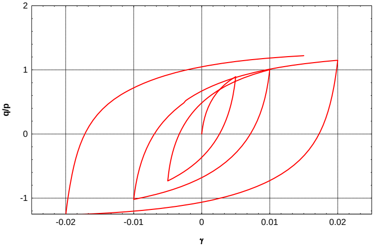


Sanisand 2004

Isobaric cyclic triaxial test



Pradhan, 1989



Sanisand 2004

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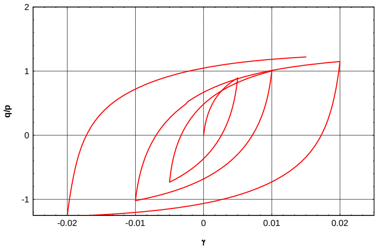
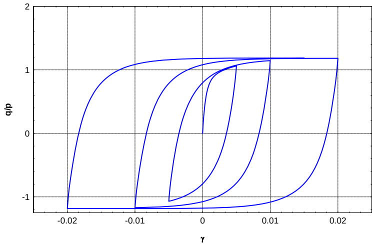
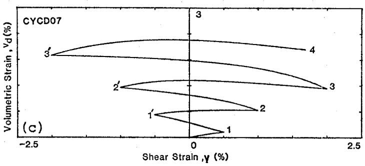


Sanisand 2004

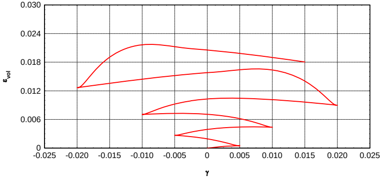


Hypoplasticity

Isobaric cyclic triaxial test

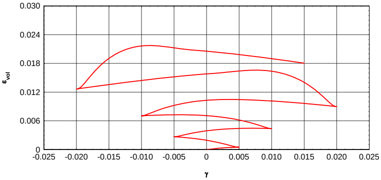


Pradhan, 1989

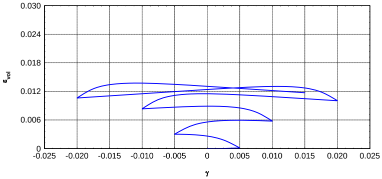


Sanisand 2004

Isobaric cyclic triaxial test

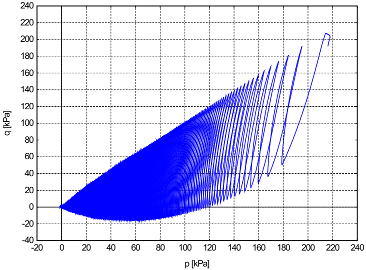


Sanisand 2004

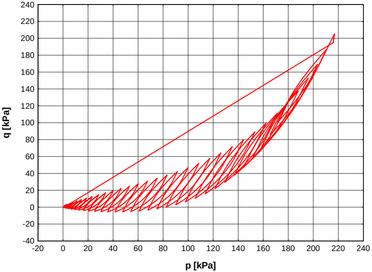


Hypoplasticity

Qualitative comparison

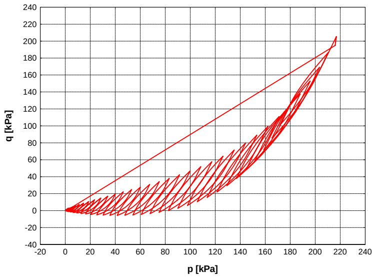


Wichtmann, 2008

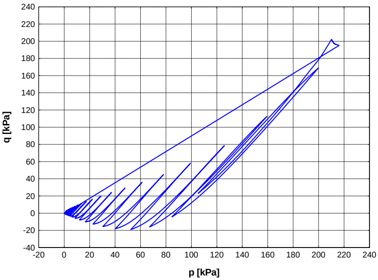


Sanisand 2004

Qualitative comparison



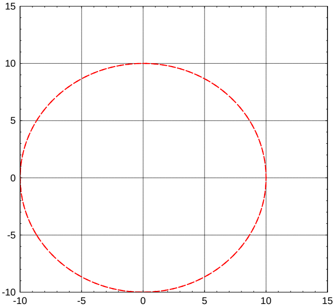
Sanisand 2004



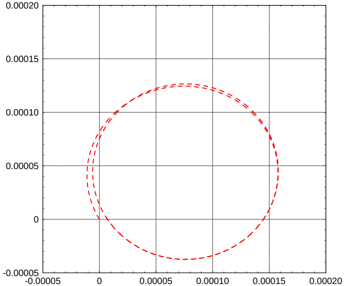
Hypoplasticity

True triaxial test

Stress controlled path - Sanisand2004



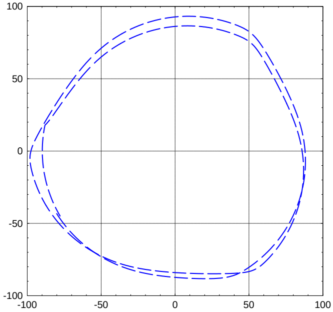
Stress path on the deviatoric plane



Strain path on the deviatoric plane

True triaxial test

Stress controlled path - Hypoplasticity



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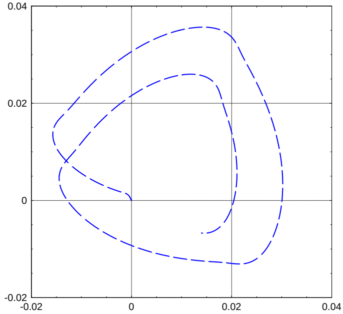
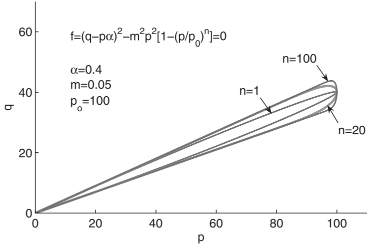


Strain path on the deviatoric plane

A critical yield surface with a cap

- SANISAND 2004 does not allow for flow under loading proportional path
- SANISAND 2007 incorporates a yield surface with a cap
- The cap causes numerical problems



Yield surface SANISAND 2007, Taiebat (2008)

Problem with the deviatoric non-associativity

In the original version the flow rule was

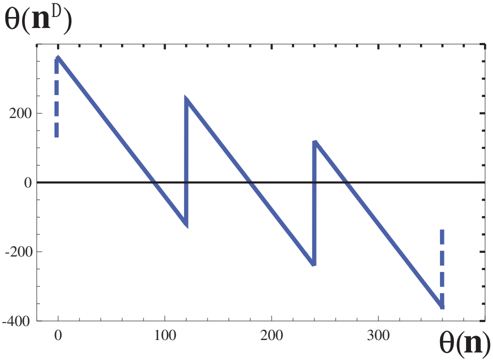
$$\mathbf{M} = \left[B\mathbf{n} - C \left(\mathbf{n}\cdot\mathbf{n} - \frac{1}{3}\,\mathbf{1} \right) + \frac{1}{3} D\,\mathbf{1} \right]^{\rightarrow} \tag{23}$$

with $C = 3\sqrt{\frac{3}{2}\frac{1-c}{c}} g$.

The tensor $\mathbf{n}^D = \left(\mathbf{n}\cdot\mathbf{n} - \frac{1}{3}\,\mathbf{1}\right)$ was supposed to introduce just a deviatoric deviation between **M** and **n** but it causes instability, e.g growth of unsymmetry in stress components $(T_{22} \neq T_{33})$

Problem with the deviatoric non-associativity

The tensor $\mathbf{n}^D$ causes instabilities because θ jumps



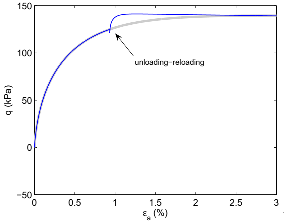
Discontinuities of the $\theta(\mathbf{n}^D)$

Unstable evolution of α

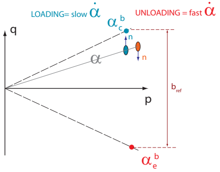
The rate of evolution of α depends on the latest reversal value of α, denoted as α_{in}.

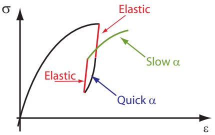
The tensor α_{in} does not evolve continuously but jumps(!) rather.

A small unloading reloading loop causes a finite change in α_{in}.



Example of overshooting. Taiebat (2008)





Concluding remark

- The implementation of any constitutive model requires a sound understanding of its theoretical aspects.
- `IncrementalDriver` is a powerful tool that allows to analyse the behaviour of the constitutive laws under different element test conditions.

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