Cosserat Continuum

Theory of micropolar elasticity

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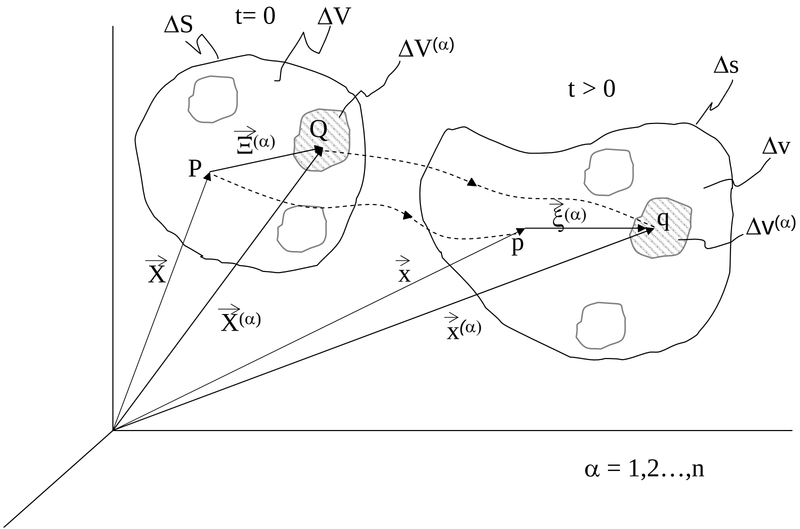


Deformation of a microvolume

An element ΔV enclosed within its surface ΔS in the undeformed body. Let the center of mass of ΔV has the position vector $\overrightarrow{X}$. Suppose that the element ΔV contains N discrete micromaterial elements $\Delta V^{(\alpha)}$, $(\alpha = 1, 2, .., N)$. The position vector of a material point in the αth microelement may be expressed as:

$$\overrightarrow{X}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi}^{(\alpha)} \quad (1)$$

Where $\overrightarrow{\Xi}^{(\alpha)}$ is the position of a point in the microelement relative to the center of mass of ΔV.

Deformation and Microdeformation

The final position of the αth particle will be:

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{x} + \overrightarrow{\xi}^{(\alpha)} \tag{2}$$

where $\overrightarrow{x}$ is the new position vector of the center of mass of Δv and $\overrightarrow{\xi}^{(\alpha)}$ is the new relative position vector of the point originally located at $\overrightarrow{X}^{(\alpha)}$.
The motion of the center of mass P of ΔV is:

$$\overrightarrow{x} = \overrightarrow{x}(\overrightarrow{X}, t) \tag{3}$$

The relative position vector $\overrightarrow{\xi}^{(\alpha)}$ depends not only on $\overrightarrow{X}$ y t but also on $\overrightarrow{\Xi}^{(\alpha)}$, i.e.,

$$\overrightarrow{\xi}^{(\alpha)} = \overrightarrow{\xi}^{(\alpha)}(\overrightarrow{X}, \overrightarrow{\Xi}^{(\alpha)}, t) \tag{4}$$

Deformation and Microdeformation

The materials points in ΔV undergo a deformation about the centre of mass.

$$\overrightarrow{\xi}^{(\alpha)} = \underline{\chi}(\overrightarrow{X}, t)\overrightarrow{\Xi}^{(\alpha)} \tag{5}$$

In coordinate form, for the spatial position of a material point in a microelement, we have:

$$x_k^{(\alpha)} = x_k(\overrightarrow{X}, t) + \chi_{kK}(\overrightarrow{X}, t)\Xi_K^{(\alpha)} \tag{6}$$

$$k, K = 1, 2, 3$$

Deformation and Microdeformation

We introduce the inverse micromotions Λ_{Kk} such that

$$\chi_{kK}\Lambda_{Kl} = \delta_{kl}, \quad \chi_{kK}\Lambda_{Lk} = \delta_{KL}, \quad \chi_{kK}\Lambda_{Lk} = \underline{1} \tag{7}$$

In component form Eq. 5 reads

$$\xi_k^{(\alpha)} = \chi_{kK}(\overrightarrow{X}, t)\Xi_K^{(\alpha)} \tag{8}$$

Using Eq. 7 in the Eq. 8, we get

$$\Xi_K^{(\alpha)} = \Lambda_{Kk}(\overrightarrow{x}, t)\xi_k^{(\alpha)} \tag{9}$$

In vector form this reads

$$\overrightarrow{\Xi}^{(\alpha)} = \underline{\Lambda}(\overrightarrow{x}, t)\,\overrightarrow{\xi}^{(\alpha)} \tag{10}$$

Deformation and Microdeformation

The motion and the inverse motion of a material point in a microelement are therefore expressed by

$$x_k^{(\alpha)} = x_k(\overrightarrow{X}, t) + \chi_{kK}(\overrightarrow{X}, t)\Xi_K^{(\alpha)} \tag{11}$$

$$X_K^{(\alpha)} = X_K(\overrightarrow{x}, t) + \Lambda_{Kk}(\overrightarrow{x}, t)\xi_k^{(\alpha)} \tag{12}$$

In vectorial form, these read

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{x}(\overrightarrow{X}, t) + \underline{\chi}(\overrightarrow{X}, t)\,\overrightarrow{\Xi}^{(\alpha)} \tag{13}$$

$$\overrightarrow{X}^{(\alpha)} = \overrightarrow{X}(\overrightarrow{x}, t) + \underline{\Lambda}(\overrightarrow{x}, t)\,\overrightarrow{\xi}^{(\alpha)} \tag{14}$$

The differential line element in the deformed body is calculated by Eq. 11. The index repeated of the last term can be changed $(K = L)$

$$x_k^{(\alpha)} = x_k(\overrightarrow{X},t) + \underbrace{\chi_{kK}(\overrightarrow{X},t)\Xi_K^{(\alpha)}}_{K=L} \quad = x_k(\overrightarrow{X},t) + \chi_{kL}(\overrightarrow{X},t)\Xi_L^{(\alpha)}$$

$$dx_k^{(\alpha)} = \frac{\partial x_k}{\partial X_K}dX_K + \chi_{kL}d\Xi_L + \Xi_L\frac{\partial \chi_{kL}}{\partial X_K}dX_K$$

$$dx_k^{(\alpha)} = \left(\frac{\partial x_k}{\partial X_K} + \Xi_L\frac{\partial \chi_{kL}}{\partial X_K}\right)dX_K + \chi_{kL}d\Xi_L \qquad (15)$$

The square of the arc length is calculated by

$$(ds^{(\alpha)})^2 = d\overrightarrow{x}^{(\alpha)} \cdot d\overrightarrow{x}^{(\alpha)} = dx_k^{(\alpha)} \cdot dx_k^{(\alpha)}$$

Using Eq 15 and forming the scalar product, we have

$$(ds^{(\alpha)})^2 = \left[\left(\frac{\partial x_k}{\partial X_K} + \underbrace{\Xi_L \frac{\partial \chi_{kL}}{\partial X_K}}_{L=M} \right) dX_K + \underbrace{\chi_{kL} d\Xi_L}_{L=K} \right] \cdot$$

$$\left[\underbrace{\left(\frac{\partial x_k}{\partial X_K} + \Xi_L \frac{\partial \chi_{kL}}{\partial X_K} \right) dX_K}_{L=N;K=L} + \chi_{kL} d\Xi_L \right]$$

Strain and Microstrain Tensors

$$(ds^{(\alpha)})^2 = \left[\left(\frac{\partial x_k}{\partial X_K} + \Xi_M \frac{\partial \chi_{kM}}{\partial X_K}\right) dX_K + \chi_{kK} d\Xi_K\right] \cdot \left[\left(\frac{\partial x_k}{\partial X_L} + \Xi_N \frac{\partial \chi_{kN}}{\partial X_L}\right) dX_L + \chi_{kL} d\Xi_L\right]$$

$$(ds^{(\alpha)})^2 = \left[\frac{\partial x_k}{\partial X_K} dX_K + \Xi_M \frac{\partial \chi_{kM}}{\partial X_K} dX_K + \chi_{kK} d\Xi_K\right] \cdot \left[\frac{\partial x_k}{\partial X_L} dX_L + \Xi_N \frac{\partial \chi_{kN}}{\partial X_L} dX_L + \chi_{kL} d\Xi_L\right]$$

On operating the inner product

$$= \underbrace{\frac{\partial x_k}{\partial X_K} dX_K \frac{\partial x_k}{\partial X_L} dX_L}_{1} + \underbrace{\frac{\partial x_k}{\partial X_K} dX_K \Xi_N \frac{\partial \chi_{kN}}{\partial X_L} dX_L}_{2} + \underbrace{\frac{\partial x_k}{\partial X_K} dX_K \chi_{kL} d\Xi_L}_{3}$$

$$+ \underbrace{\Xi_M \frac{\partial \chi_{kM}}{\partial X_K} dX_K \frac{\partial x_k}{\partial X_L} dX_L}_{4} + \underbrace{\Xi_M \frac{\partial \chi_{kM}}{\partial X_K} dX_K \Xi_N \frac{\partial \chi_{kN}}{\partial X_L} dX_L}_{5} +$$

$$\underbrace{\Xi_M \frac{\partial \chi_{kM}}{\partial X_K} dX_K \chi_{kL} d\Xi_L}_{6} + \underbrace{\chi_{kK} d\Xi_K \frac{\partial x_k}{\partial X_L} dX_L}_{7} + \underbrace{\chi_{kK} d\Xi_K \Xi_N \frac{\partial \chi_{kN}}{\partial X_L} dX_L}_{8} +$$

$$\underbrace{\chi_{kL} d\Xi_L \chi_{kK} d\Xi_K}_{9}$$

Joining (1-2-4-5), (3-6), (7-8) and factorizing

$$\left(\frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L}+\frac{\partial x_k}{\partial X_K}\underbrace{\Xi_N\frac{\partial \chi_{kN}}{\partial X_L}}_{N=M}+\underbrace{\frac{\partial \chi_{kM}}{\partial X_K}\frac{\partial x_k}{\partial X_L}\Xi_M}_{K=L,L=K}+\Xi_M\frac{\partial \chi_{kM}}{\partial X_K}\Xi_N\frac{\partial \chi_{kN}}{\partial X_L}\right)dX_K dX_L$$

$$+\chi_{kL}\chi_{kK}d\Xi_L d\Xi_K+\left(\frac{\partial x_k}{\partial X_K}\chi_{kL}+\Xi_M\frac{\partial \chi_{kM}}{\partial x_k}\chi_{kL}\right)dX_K d\Xi_L+$$
$$\underbrace{\left(\chi_{kK}\frac{\partial x_k}{\partial X_L}+\chi_{kK}\underbrace{\Xi_N\frac{\partial \chi_{kN}}{\partial X_L}}_{N=M}\right)d\Xi_K dX_L}_{K=L,L=K}$$

$$(ds^{(\alpha)})^2 = (\frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L} + \frac{\partial x_k}{\partial X_K}\Xi_M\frac{\partial \chi_{kM}}{\partial X_L} + \frac{\partial \chi_{kM}}{\partial X_K}\frac{\partial x_k}{\partial X_L}\Xi_M +$$

$$\Xi_M\frac{\partial \chi_{kM}}{\partial X_K}\Xi_N\frac{\partial \chi_{kN}}{\partial X_L})dX_K dX_L + \chi_{kL}\chi_{kK}d\Xi_L d\Xi_K + (\frac{\partial x_k}{\partial X_K}\chi_{kL} +$$

$$\Xi_M\frac{\partial \chi_{kM}}{\partial x_k}\chi_{kL})dX_K d\Xi_L + (\chi_{kL}\frac{\partial x_k}{\partial X_K} + \chi_{kL}\Xi_M\frac{\partial \chi_{kM}}{\partial X_K})d\Xi_L dX_K$$

$$(ds^{(\alpha)})^2 = (\underbrace{\frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L}}_{C_{KL}} + 2\underbrace{\frac{\partial x_k}{\partial X_K}\frac{\partial \chi_{kM}}{\partial X_L}}_{\Gamma_{KML}}\Xi_M + \Xi_M\frac{\partial \chi_{kM}}{\partial X_K}\Xi_N\frac{\partial \chi_{kN}}{\partial X_L})dX_K dX_L +$$

$$2(\underbrace{\frac{\partial x_k}{\partial X_K}\chi_{kL}}_{\Psi_{KL}} + \chi_{kL}\frac{\partial \chi_{kM}}{\partial X_K})dX_K d\Xi_L + \chi_{kL}\chi_{kK}d\Xi_L d\Xi_k$$

Strain and Microstrain Tensors

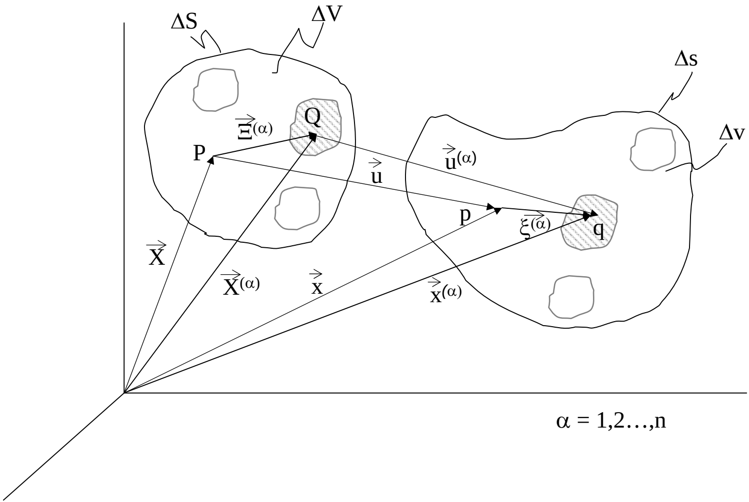
$$(ds^{(\alpha)})^2 = \left[C_{KL} + 2\Gamma_{KML}\Xi_M + \Xi_M \frac{\partial \chi_{kM}}{\partial X_K} \Xi_N \frac{\partial \chi_{kN}}{\partial X_L}\right] dX_K dX_L + 2\left[\Psi_{KL} + \chi_{kL} \frac{\partial \chi_{kM}}{\partial X_K}\right] dX_K d\Xi_L + \chi_{kL}\chi_{kK} d\Xi_L d\Xi_k$$

Where:

$$C_{KL}(\overrightarrow{X}, t) \equiv \frac{\partial x_k}{\partial X_K} \frac{\partial x_k}{\partial X_L} \quad (16)$$

$$\Psi_{KL}(\overrightarrow{X}, t) \equiv \frac{\partial x_k}{\partial X_K} \chi_{kL} \quad (17)$$

$$\Gamma_{KLM}(\overrightarrow{X}, t) \equiv \frac{\partial x_k}{\partial X_K} \frac{\partial \chi_{kM}}{\partial X_L} \quad (18)$$



Displacement vectors

Strain and Microstrain Tensors

From the last figure:

$$\overrightarrow{X} + \overrightarrow{U} = \overrightarrow{x} \Rightarrow \quad \overrightarrow{U} = \overrightarrow{x} - \overrightarrow{X} \tag{19}$$

$$\Xi^{(\alpha)} + \overrightarrow{U}^{(\alpha)} = \overrightarrow{U} + \overrightarrow{\xi}^{(\alpha)}$$

$$\overrightarrow{U}^{(\alpha)} = \overrightarrow{U} + \overrightarrow{\xi}^{(\alpha)} - \overrightarrow{\Xi}^{(\alpha)} \tag{20}$$

$$x_K = U_K + X_K \tag{21}$$

Strain and Microstrain Tensors

From Eq 21 , by partial differentiation, we obtain

$$\frac{\partial x_k}{\partial X_K} = \frac{\partial U_k}{\partial X_K} + \frac{\partial X_k}{\partial X_K} = \frac{\partial U_k}{\partial X_K} + \delta_{kK} \qquad (22)$$

Similarly, we introduce the microdisplacement tensors $\Phi_{LK}(\overrightarrow{X}, t)$

$$\chi_{kK} = (\delta_{LK} + \Phi_{LK})\delta_{kL} = \delta_{LK}\delta_{kL} + \Phi_{LK}\delta_{kL} = \delta_{kK} + \Phi_{kK} \qquad (23)$$

By use of Eqs. 19, 23 the Eq. 20 may also expressed as

$$\overrightarrow{U}^{(\alpha)} = \overrightarrow{U} + \overrightarrow{\xi} - \overrightarrow{\Xi} = \overrightarrow{U} + \underline{\chi}\,\overrightarrow{\Xi} - \overrightarrow{\Xi} = \overrightarrow{U} + \overrightarrow{\Xi}(\underline{\chi} - \underline{1})$$

$$U_K^{(\alpha)} = U_K + \Xi_K(\delta_{kK} + \Phi_{kK} - \underline{1}) = U_K + \Xi_K(\delta_{kK} + \Phi_{kK} - \delta_{kK})$$

$$U_K^{(\alpha)} = U_K + \Xi_K \Phi_{kK} \qquad (24)$$

Strain and Microstrain Tensors

On substituing Eqs. 22 and 23 into Eqs. 16 - 18 we find that

$$\begin{aligned} C_{KL} &= \frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L} = \left(\frac{\partial U_k}{\partial X_K} + \delta_{kK}\right)\left(\frac{\partial U_k}{\partial X_L} + \delta_{kL}\right) \\ &= \underbrace{\frac{\partial U_k}{\partial X_K}\frac{\partial U_k}{\partial X_L}}_{\approx 0} + \underbrace{\frac{\partial U_k}{\partial X_K}\delta_{kL}}_{k=L} + \underbrace{\delta_{kK}\frac{\partial U_k}{\partial X_L}}_{k=L} + \underbrace{\delta_{kK}\delta_{kL}}_{k=L} \end{aligned} \tag{25}$$

$$\begin{aligned} \Psi_{KL} &= \frac{\partial x_k}{\partial X_K}\chi_{kL} = \left(\frac{\partial U_k}{\partial X_K} + \delta_{kK}\right)\left(\delta_{kL} + \Phi_{kL}\right) \\ \Psi_{KL} &= \left(\underbrace{\frac{\partial U_k}{\partial X_K}\delta_{kL}}_{k=L} + \underbrace{\frac{\partial U_k}{\partial X_K}\Phi_{kL}}_{\approx 0} + \underbrace{\delta_{kK}\delta_{kL}}_{k=L} + \delta_{kK}\Phi_{kL}\right) \end{aligned} \tag{26}$$

$$\Gamma_{KLM} = \frac{\partial x_k}{\partial X_K}\frac{\partial \chi_{kM}}{\partial X_L} = \left(\frac{\partial U_k}{\partial X_K} + \delta_{kK}\right)\left(\frac{\partial(\delta_{kM} + \Phi_{kM})}{\partial X_L}\right)$$

$$= \left(\frac{\partial U_k}{\partial X_K} + \delta_{kK}\right)\left(\frac{\partial \Phi_{kM}}{\partial X_L}\right) = \underbrace{\frac{\partial U_k}{\partial X_K}\frac{\partial \Phi_{KM}}{\partial X_L}}_{\approx 0} + \underbrace{\delta_{kK}\frac{\partial \Phi_{kM}}{\partial X_L}}_{k=K} \quad (27)$$

These expressions are exact. For a linear theory, it is assumes that the product terms are negligible.

$$C_{KL} \approx \frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L} + \delta_{KL} \quad (28)$$

$$\Psi_{KL} \approx \frac{\partial U_L}{\partial X_K} + \delta_{KL} + \Phi_{KL} \quad (29)$$

Strain and Microstrain Tensors

$$\Gamma_{KLM} \approx \frac{\partial \Phi_{KM}}{\partial X_L} \quad (30)$$

The difference between the spatial and material representation disappears, so that one may use u_k in place of U_K and ϕ_{kl} in place of Φ_{KL}, etc.
The material strain tensor E_{KL} and the material microstrain tensors $\mathscr{E}_{KL}$ and Γ_{KLM} are defined thus:

$$E_{KL} \equiv \frac{1}{2}(C_{KL} - \delta_{KL}) = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L} + \delta_{KL} - \delta_{KL}\right) \quad (31)$$

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L}\right) \quad (32)$$

$$\mathscr{E}_{KL} \equiv \Psi_{KL} - \delta_{KL} = \frac{\partial U_L}{\partial X_K} + \delta_{KL} + \Phi_{KL} - \delta_{KL} = \frac{\partial U_L}{\partial X_K} + \Phi_{KL} \quad (33)$$

$$\Gamma_{KLM} \equiv \frac{\partial \Phi_{KM}}{\partial X_L} \quad (34)$$

Micropolar Strains and Rotations

The tensor Φ_{KL} is defined as antisymmetric for the theory of micropolar elasticity.

$$\Phi_{KL} = -\Phi_{LK} \tag{35}$$

in the spatial notation, $\phi_{kl} = -\phi_{lk}$. Every skew-symmetric, second tensor Φ_{KL} can be expressed by an axial vector Φ_K defined by:

$$\Phi_K = \frac{1}{2} e_{KLM} \Phi_{ML} \tag{36}$$

Eq 36 can be expressed as:

$$\Phi_1 = \Phi_{32}, \quad \Phi_2 = \Phi_{13}, \quad \Phi_3 = \Phi_{21}$$

The Eq 36 can be written as

$$\Phi_{KL} = -e_{KLM} \Phi_M \tag{37}$$

It is known that:

$$\begin{aligned}
\underline{\Phi}\overrightarrow{U} &= \overrightarrow{\Phi} \times \overrightarrow{U} \\
(\Phi_{LK}U_K)_{\hat{e_L}} &= \Phi_L\hat{e_L} \times U_K\hat{e_K} \\
(\Phi_{LK}U_K)_{\hat{e_L}} &= (\Phi_L U_K e_{LKM})\hat{e_M}
\end{aligned}$$

It is changed the subindex in order to have the components in the same direction

$$\begin{aligned}
(\Phi_{LK}U_K)_{\hat{e_L}} &= (\Phi_M U_K e_{MKL})\hat{e_L} \\
\Phi_{LK} &= \Phi_M e_{MKL}
\end{aligned}$$

$\underline{\Phi}$ is an antysimmetric tensor, then $\Phi_{LK} = -\Phi KL$

$$\begin{aligned}
-\Phi_{KL} &= \Phi_M e_{MKL} \\
\Phi_{KL} &= -e_{MKL}\Phi_M \\
\Phi_{KL} &= -e_{KLM}\Phi_M
\end{aligned}$$

That is the proof of Equation 37

The Equation 36 is proved beginning with the Equation 37:

$$\Phi_{KL} = -e_{KLM}\Phi_M$$

Multiplying to both sides of the equation by e_{RLK}

$$\begin{aligned} e_{RLK}\Phi_{KL} &= -e_{KLM}e_{RLK}\Phi_M \\ e_{RLK}\Phi_{KL} &= -e_{KLM} - e_{RKL}\Phi_M \\ e_{RLK}\Phi_{KL} &= e_{KLM}e_{RKL}\Phi_M \\ e_{RLK}\Phi_{KL} &= e_{MKL}e_{RKL}\Phi_M \end{aligned}$$

From identities between permutation symbol and Kronecker delta

$$\begin{aligned} e_{RLK}\Phi_{KL} &= 2\delta_{MR}\Phi_M \\ \frac{e_{RLK}\Phi_{KL}}{2} &= \delta_{MR}\Phi_M \\ \frac{1}{2}e_{RLM}\Phi_{ML} &= \delta_{KR}\Phi_R \end{aligned}$$

On multiplying both sides of the last equation by δ_{KR}

$$\begin{aligned} \underbrace{\delta_{KR}\frac{1}{2}e_{RLM}\Phi_{ML}}_{R=K} &= \underbrace{\delta_{KR}\delta_{KR}\Phi_{R}}_{R=K} \\ \frac{1}{2}e_{KLM}\Phi_{ML} &= \Phi_{K} \end{aligned}$$

Micropolar Strains and Rotations

Substituting this into Eq 23, we have

$$\chi_{kK} = \delta_{kK} + \Phi_{kK} = \delta_{kK} - e_{kKM}\Phi_M \tag{38}$$

In the classical theory, we have the rotational tensor

$$R_{KL} = -R_{LK} \equiv \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} - \frac{\partial U_L}{\partial X_K}\right) \tag{39}$$

The axial vector R_K

$$R_K = \frac{1}{2} e_{KLM} R_{ML} \quad R_{KL} = -e_{KLM} R_M \tag{40}$$

Micropolar Strains and Rotations

Using Eqs. 31 y 40, we obtain

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L}\right) \quad R_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} - \frac{\partial U_L}{\partial X_K}\right)$$

$$\frac{\partial U_K}{\partial X_L} = E_{KL} + R_{KL}$$

$$\frac{\partial U_K}{\partial X_L} = E_{KL} - e_{KLM} R_M \tag{41}$$

$$\frac{\partial U_L}{\partial X_K} = E_{LK} - e_{LKM} R_M = E_{KL} + e_{KLM} R_M$$

Where: $E_{KL} = E_{LK}$ and $e_{LKM} = -e_{KLM}$

When this and Eq. 37 are substituted into Eq. 33 and 34 we get

$$\mathscr{E}_{KL} \equiv \frac{\partial U_L}{\partial X_K} + \Phi_{KL} = E_{KL} + e_{KLM} R_M - e_{KLM} \Phi_M$$

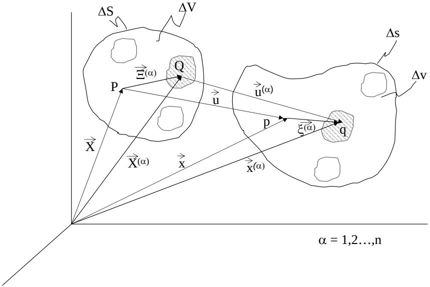
$$\mathscr{E}_{KL} = E_{KL} + e_{KLM}(R_M - \Phi_M) \quad (42)$$

$$\Gamma_{KLM} = \frac{\partial \Phi_{KL}}{\partial X_M} = \frac{\partial(-e_{KLN}\Phi_N)}{\partial X_M} = -e_{KLN}\frac{\partial \Phi_N}{\partial X_M} \quad (43)$$

When $R_M = \Phi_M$, we see that $\varepsilon_{KL} = E_{KL}$ and $\Gamma_{KLM} = \dfrac{\partial R_{KL}}{\partial X_M}$ and the microstrains are not independent of the classical strain and rotations. The micropolar theory assumes that the classical rotation R_K is different from the microrotation.

Micropolar Strains and Rotations

Regarding this figure (Displacement vectors) and the Eq. 24:



Displacement Vectors

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi} + \overrightarrow{U}^{(\alpha)} \qquad U_K^{(\alpha)} = U_K + \Phi_{kK}\Xi_K$$

$$\underline{\Phi}\,\overrightarrow{\Xi} = \overrightarrow{\Phi} \times \overrightarrow{\Xi}$$

$\underline{\Phi}$ is antisymmetric.

Micropolar Strains and Rotations

The spatial position of the αth point $\overrightarrow{x}^{(\alpha)}$ is obtained

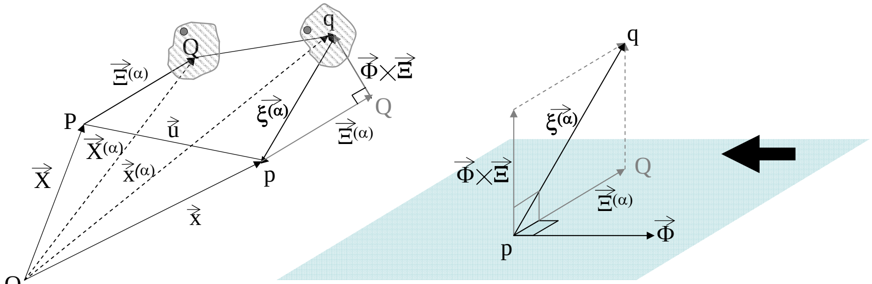
$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi} + \overrightarrow{U}^{(\alpha)}$$

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi} + \overrightarrow{U} + \underline{\Phi}\,\overrightarrow{\Xi}$$

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi} + \overrightarrow{U} + \overrightarrow{\Phi} \times \overrightarrow{\Xi}$$

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{U} + \underbrace{\overrightarrow{\Xi} + \overrightarrow{\Phi} \times \overrightarrow{\Xi}}_{\overrightarrow{\xi}}$$

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{X} + \overrightarrow{\Xi} + \overrightarrow{U} - \overrightarrow{\Xi} \times \overrightarrow{\Phi} \tag{44}$$



Microrotations

$$\vec{x}^{(\alpha)} = \vec{X} + \vec{U} + \underbrace{\vec{\Xi} + \vec{\Phi} \times \vec{\Xi}}_{\vec{\xi}}$$

Micropolar Strains and Rotations

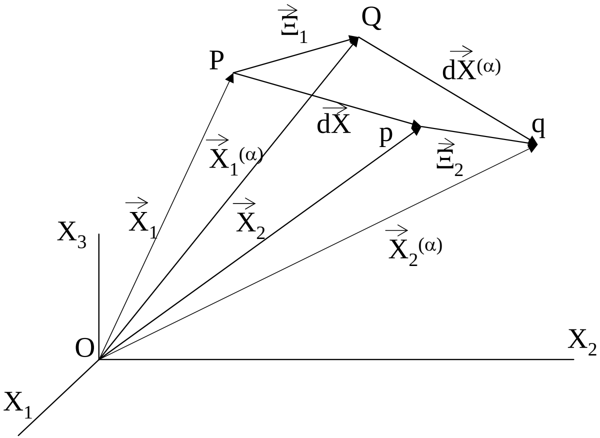
$\overrightarrow{\Phi}$ represents an angular rotation of a microelement about the center of mass of the deformed macrovolume element.

$$\overrightarrow{\xi} = \overrightarrow{\Xi} - \overrightarrow{\Xi} \times \overrightarrow{\Phi} \tag{45}$$

Which shows the relative position $\overrightarrow{\Xi}$ of a material point after deformation is obtained by translating $\overrightarrow{\Xi}$ parallel to itself to the center of mass $\overrightarrow{x}$ of the macrovolume element and then rotating it in accordance with $\overrightarrow{\Phi} \times \overrightarrow{\Xi}$.

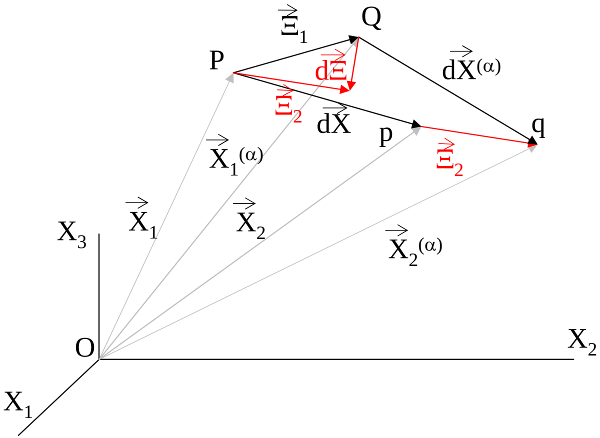
Micropolar Strains and Rotations

Consider the deformation of an infinitesimal vector $\overrightarrow{dX}^{(\alpha)} = \overrightarrow{dX} + d\overrightarrow{\Xi}$ at $\overrightarrow{X} + \overrightarrow{\Xi}$.



Deformation of an infinitesimal vector

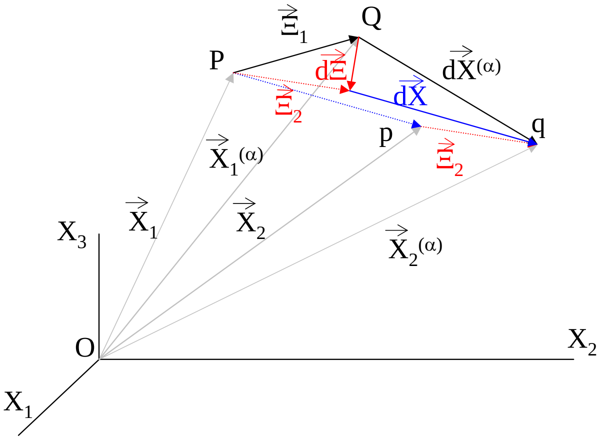
Consider the deformation of an infinitesimal vector $\overrightarrow{dX}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi}$ at $\overrightarrow{X} + \overrightarrow{\Xi}$.



Deformation of an infinitesimal vector

Micropolar Strains and Rotations

Consider the deformation of an infinitesimal vector $\overrightarrow{dX}^{(\alpha)} = \overrightarrow{dX} + d\overrightarrow{\Xi}$ at $\overrightarrow{X} + \overrightarrow{\Xi}$.



Deformation of an infinitesimal vector

Micropolar Strains and Rotations

On deformation, this vector becomes: (Derivating Eq. 44)

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} + \frac{\partial \overrightarrow{U}}{\partial X_K} dX_K - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} - \overrightarrow{\Xi} \times \frac{\partial \overrightarrow{\Phi}}{\partial X_K} dX_K \qquad (46)$$

By use of Eq. 41 and 40 we may write

$$\frac{\partial \overrightarrow{U}}{\partial X_K} dX_K = \frac{\partial U_L}{\partial X_K} dX_K = E_{LK} dX_K + \underbrace{R_{LK}}_{Antisymmetric} dX_K$$

$$\frac{\partial \overrightarrow{U}}{\partial X_K} dX_K = E_{KL} dX_K + \overrightarrow{R} \times d\overrightarrow{X} = E_{KL} dX_K - d\overrightarrow{X} \times \overrightarrow{R} \qquad (47)$$

Micropolar Strains and Rotations

Similarly, using Eq. 43

$$\begin{aligned}\overrightarrow{\Xi} \times \frac{\partial \overrightarrow{\Phi}}{\partial X_N} dX_N &= -\frac{\partial \overrightarrow{\Phi}}{\partial X_N} \times \overrightarrow{\Xi}\, dX_N = -\frac{\partial \underline{\Phi}}{\partial X_N} \overrightarrow{\Xi}\, dX_N \\ &= e_{KLM} \frac{\partial \Phi_M}{\partial X_N} \Xi_L dX_N = \Gamma_{KLN} \Xi_L dX_N\end{aligned}$$

For convenience, we introduce the notation

$$\Gamma_{KM} \equiv \Gamma_{KLM} \Xi_L \qquad (48)$$

so that

$$\begin{aligned}\overrightarrow{\Xi} \times \frac{\partial \overrightarrow{\Phi}}{\partial X_N} dX_N &= \Gamma_{KM} dX_M = \left(\Gamma_{(KM)} + \Gamma_{[KM]}\right) dX_M \\ &= \underbrace{\Gamma_{(KM)}}_{Symmetric} dX_M + \underbrace{\Gamma_{[KM]}}_{Antisymmetric} dX_M \qquad (49)\end{aligned}$$

Micropolar Strains and Rotations

Carrying Eqs 47 y 49 into Eq. 46, we rearrange it as:

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} + (E_{KL} dX_k - \overrightarrow{dX} \times \overrightarrow{R}) - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \underbrace{\Gamma_{(KM)} dX_M}_{K=M;M=K} + \underbrace{\Gamma_{[KM]} dX_M}_{\overrightarrow{\Gamma} \times \overrightarrow{dX}}$$

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} dX_K + \underbrace{\Gamma_{(MK)}}_{symmetric} dX_K)$$

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} + \Gamma_{(KM)}) dX_K \quad (50)$$

Now, it is defined a new minirotation vector $\vec{\Gamma}$ by

$$\Gamma_K \equiv \frac{1}{2} e_{KLM} \Gamma_{ML}, \quad \Gamma_{[KL]} = -e_{KLM} \Gamma_M \tag{51}$$

This vector is called minirotation for distinction from the microrotation Φ. Eq.51 can be written:

$$\Gamma_K = \frac{1}{2} \left(-\frac{\partial \Phi_L}{\partial X_L} \Xi_K + \frac{\partial \Phi_K}{\partial X_L} \Xi_L \right) \tag{52}$$

Eq. 52 is obtained of the following manner:

Micropolar Strains and Rotations

$$\Gamma_K = \frac{1}{2} e_{KLM} \Gamma_{ML}$$

From de Eq. 48 $\Gamma_{KM} = \Gamma_{KLM} \Xi_L \rightarrow \Gamma_{ML} = \Gamma_{MNL} \Xi_N$

$$\Gamma_K = \frac{1}{2} e_{KLM} \Gamma_{MNL} \Xi_N$$

From Eq 43 $\Gamma_{KLM} = -e_{KLN} \frac{\partial \Phi_N}{\partial X_M} \rightarrow \Gamma_{MNL} = -e_{MNP} \frac{\partial \Phi_P}{\partial X_L}$

$$\Gamma_K = -\frac{1}{2} e_{MKL} e_{MNP} \frac{\partial \Phi_P}{\partial X_L} \Xi_N$$

Using identities of permutation and Kronecker delta is obtained:

$$\Gamma_K = -\frac{1}{2} (\delta_{KN} \delta_{LP} - \delta_{KP} \delta_{LN}) \frac{\partial \Phi_P}{\partial X_L} \Xi_N$$

Micropolar Strains and Rotations

$$\Gamma_K = -\frac{1}{2}\left(\underbrace{\delta_{KN}\delta_{LP}\frac{\partial \Phi_P}{\partial X_L}\Xi_N}_{N=K;P=L} - \underbrace{\delta_{KP}\delta_{LN}\frac{\partial \Phi_P}{\partial X_L}\Xi_N}_{P=K;N=L}\right)$$

$$\Gamma_K = -\frac{1}{2}\left(\frac{\partial \Phi_L}{\partial X_L}\Xi_K - \frac{\partial \Phi_k}{\partial X_L}\Xi_L\right)$$

$$\Gamma_K = \frac{1}{2}\left(-\frac{\partial \Phi_L}{\partial X_L}\Xi_K + \frac{\partial \Phi_k}{\partial X_L}\Xi_L\right)$$

Macrostrain tensor (Eq. 31)

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_L}{\partial X_K} + \frac{\partial U_K}{\partial X_L}\right)$$

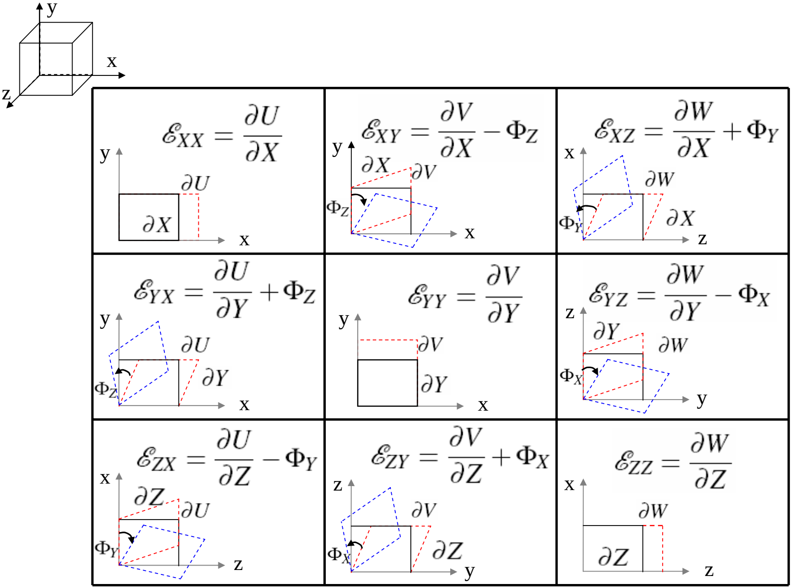
$$\begin{aligned} E_{XX} &= \frac{\partial U}{\partial X} & E_{XY} &= \frac{1}{2}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) \\ E_{YY} &= \frac{\partial V}{\partial Y} & E_{YZ} &= \frac{1}{2}\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right) \\ E_{ZZ} &= \frac{\partial W}{\partial Z} & E_{XY} &= \frac{1}{2}\left(\frac{\partial W}{\partial X} + \frac{\partial U}{\partial Z}\right) \end{aligned}$$

Micropolar Strains and Rotations

Micropolar strain tensor (Eq. 33)

$$\mathscr{E}_{KL} = \frac{\partial U_L}{\partial X_K} + \Phi_{KL} \quad \Phi_{KL} = -e_{KLM}\Phi_M \rightarrow \mathscr{E}_{KL} = \frac{\partial U_L}{\partial X_K} - e_{KLM}\Phi_M$$

$\mathscr{E}_{XX} = \frac{\partial U}{\partial X}$	$\mathscr{E}_{XY} = \frac{\partial V}{\partial X} - \Phi_Z$	$\mathscr{E}_{XZ} = \frac{\partial W}{\partial X} + \Phi_Y$
$\mathscr{E}_{YX} = \frac{\partial U}{\partial Y} + \Phi_Z$	$\mathscr{E}_{YY} = \frac{\partial V}{\partial Y}$	$\mathscr{E}_{YZ} = \frac{\partial W}{\partial Y} - \Phi_X$
$\mathscr{E}_{ZX} = \frac{\partial U}{\partial Z} - \Phi_Y$	$\mathscr{E}_{ZY} = \frac{\partial V}{\partial Z} + \Phi_X$	$\mathscr{E}_{ZZ} = \frac{\partial W}{\partial Z}$



Microdeformation Tensor

Micropolar Strains and Rotations

Micropolar strain tensor of third order (Eq. 34)

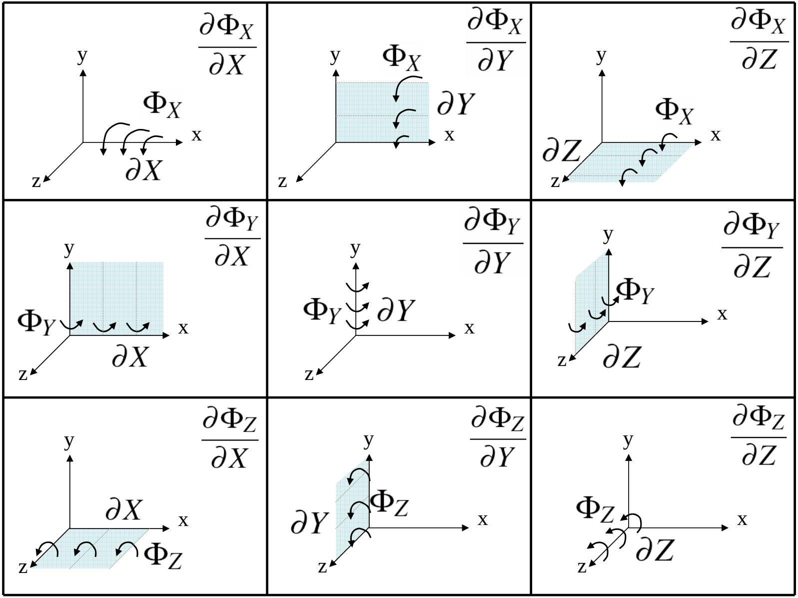
$$\Gamma_{KLM} = \frac{\partial \Phi_{KL}}{\partial X_M} \quad \Phi_{KL} = -e_{KLN}\Phi_N \rightarrow \Gamma_{KLM} = \frac{\partial(-e_{KLN}\Phi_N)}{\partial X_M} \tag{53}$$

$$\Gamma_{YZX} = -\Gamma_{ZYX} = \frac{\partial \Phi_X}{\partial X}; \Gamma_{YZY} = -\Gamma_{ZYY} = \frac{\partial \Phi_X}{\partial Y}; \Gamma_{YZZ} = -\Gamma_{ZYZ} = \frac{\partial \Phi_X}{\partial Z}$$

$$\Gamma_{ZXX} = -\Gamma_{XZX} = \frac{\partial \Phi_Y}{\partial X}; \Gamma_{ZXY} = -\Gamma_{XZY} = \frac{\partial \Phi_Y}{\partial Y}; \Gamma_{ZXZ} = -\Gamma_{XZZ} = \frac{\partial \Phi_Y}{\partial Z}$$

$$\Gamma_{XYX} = -\Gamma_{YXX} = \frac{\partial \Phi_Z}{\partial X}; \Gamma_{XYY} = -\Gamma_{YXY} = \frac{\partial \Phi_Z}{\partial Y}; \Gamma_{XYZ} = -\Gamma_{YXZ} = \frac{\partial \Phi_Z}{\partial Z}$$

all other $\Gamma_{KLM} = 0$



Micropolar strain tensor of third order

Macrorotation vector (Eq. 40)

$$R_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} - \frac{\partial U_L}{\partial X_K}\right)$$

$$R_X = \frac{1}{2}\left(\frac{\partial W}{\partial Y} - \frac{\partial V}{\partial Z}\right); R_Y = \frac{1}{2}\left(\frac{\partial U}{\partial Z} - \frac{\partial W}{\partial X}\right); \quad R_Z = \frac{1}{2}\left(\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y}\right)$$

Microrotation vector

$$\overrightarrow{\Phi} = \Phi_X \hat{I} + \Phi_Y \hat{J} + \Phi_Z \hat{K}$$

Minirotation vector (Eq. 52)

$$\Gamma_X = \frac{1}{2}\left[-\left(\frac{\partial \Phi_Y}{\partial Y} + \frac{\partial \Phi_Z}{\partial Z}\right)\Xi_X + \frac{\partial \Phi_X}{\partial Y}\Xi_Y + \frac{\partial \Phi_X}{\partial Z}\Xi_Z\right]$$

$$\Gamma_Y = \frac{1}{2}\left[-\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Z}{\partial Z}\right)\Xi_Y + \frac{\partial \Phi_Y}{\partial X}\Xi_X + \frac{\partial \Phi_Y}{\partial Z}\Xi_Z\right]$$

$$\Gamma_Z = \frac{1}{2}\left[-\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Y}{\partial Y}\right)\Xi_Z + \frac{\partial \Phi_Z}{\partial X}\Xi_X + \frac{\partial \Phi_Z}{\partial Y}\Xi_Y\right]$$

Other way to see the minirotation vector is the tensorial form $\overrightarrow{\Gamma} = \frac{1}{2}\underline{B}\overrightarrow{\Xi}$:

$$\overrightarrow{\Gamma} = \begin{bmatrix} \Gamma_X \\ \Gamma_Y \\ \Gamma_Z \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} -\left(\frac{\partial \Phi_Y}{\partial Y} + \frac{\partial \Phi_Z}{\partial Z}\right) & \frac{\partial \Phi_X}{\partial Y} & \frac{\partial \Phi_X}{\partial Z} \\ \frac{\partial \Phi_Y}{\partial X} & -\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Z}{\partial Z}\right) & \frac{\partial \Phi_Y}{\partial Z} \\ \frac{\partial \Phi_Z}{\partial X} & \frac{\partial \Phi_Z}{\partial Y} & -\left(\frac{\partial \Phi_X}{\partial X} + \frac{\partial \Phi_Y}{\partial Y}\right) \end{bmatrix}$$

$$\overrightarrow{\Xi} = \begin{bmatrix} \Xi_X \\ \Xi_Y \\ \Xi_Z \end{bmatrix}$$

Geometrical meaning of Micropolar strains

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} + \Gamma_{(KM)}) dX_K$$

Eq 50 reveals that the deformation of the vector $\overrightarrow{dX}^{(\alpha)} \equiv \overrightarrow{dX} + \overrightarrow{d\Xi}$ may be achieved by the following three operations:

1. A rigid translation of $\overrightarrow{dX} + \overrightarrow{d\Xi}$ from material centroid $\overrightarrow{X}$ to the spatial centroid $\overrightarrow{x}$.
2. Rigid rotations of $\overrightarrow{dX}$ and $\overrightarrow{d\Xi}$ by the amounts $d\overrightarrow{X} \times (\overrightarrow{R} + \overrightarrow{\Gamma})$ and $\overrightarrow{d\Xi} \times \overrightarrow{\Phi}$, respectively.
3. Finally, a stretch represented by the strains E_{KL} and $\Gamma_{(KL)}$

Geometrical meaning of Micropolar strains

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dX} + \overrightarrow{d\Xi} - (\overrightarrow{dX} \times \overrightarrow{R} + \overrightarrow{d\Xi} \times \overrightarrow{\Phi} + \overrightarrow{dX} \times \overrightarrow{\Gamma}) + (E_{KL} + \Gamma_{(KM)})dX_K$$

Eq. 50 can be written as

$$\overrightarrow{dx}^{(\alpha)} = \overrightarrow{dx} + \overrightarrow{dy} + \overrightarrow{dz}$$

Where

$$\overrightarrow{dx} = \overrightarrow{dX} - \overrightarrow{dX} \times \overrightarrow{R} + E_{KL} dX_K \tag{54}$$

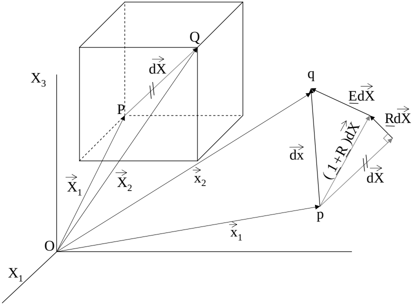
$$\overrightarrow{dy} = -\overrightarrow{dX} \times \overrightarrow{\Gamma} + \Gamma_{(KM)} dX_K \tag{55}$$

$$\overrightarrow{dz} = d\,\overrightarrow{\Xi} - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} \tag{56}$$

Geometrical meaning of Micropolar strains and Rotations

Eq. 54 can be expressed as

$$\overrightarrow{dx} = \overrightarrow{dX} + \overrightarrow{R} \times \overrightarrow{dX} + E_{KL} dX_K = \overrightarrow{dX} + \underline{R}\overrightarrow{dX} + \underline{E}\overrightarrow{dX} = \overrightarrow{dX}[\underline{1} + \underline{R}] + \underline{E}\overrightarrow{dX}$$

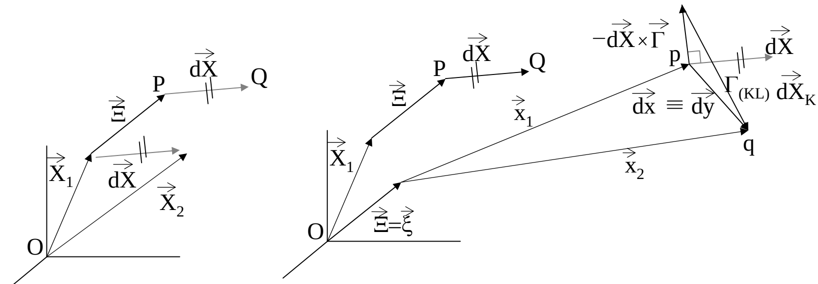


Conventional Continnum

Geometrical meaning of Micropolar strains and Rotations

Eq. 55 can be written as

$$\overrightarrow{dy} = -\overrightarrow{dX} \times \overrightarrow{\Gamma} + \Gamma_{(KL)}\overrightarrow{dX} = \Gamma_{[KL]}\overrightarrow{dX} + \Gamma_{(KL)}\overrightarrow{dX}$$

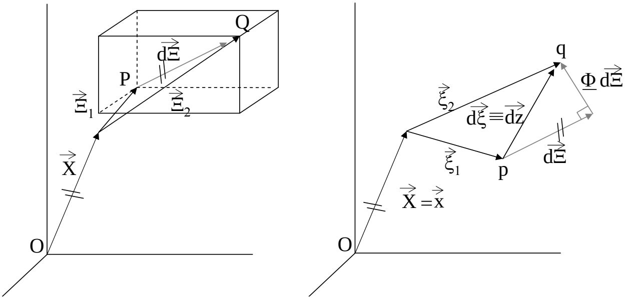


Ξ is constant

Geometrical meaning of Micropolar strains and Rotations

Eq. 56 can be expressed as

$$\overrightarrow{dz} = d\,\overrightarrow{\Xi} - \overrightarrow{d\Xi} \times \overrightarrow{\Phi} = d\,\overrightarrow{\Xi} + \overrightarrow{\Phi} \times \overrightarrow{d\Xi} = \underbrace{d\,\overrightarrow{\Xi} + \underline{\Phi}\overrightarrow{d\Xi}}_{d\,\overrightarrow{\Xi}\,(\underline{1}+\underline{\Phi})}$$



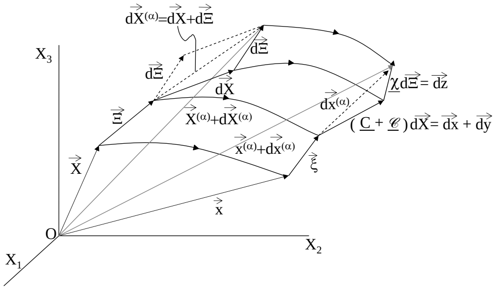
X is a constant

Geometrical meaning of Micropolar strains and Rotations

$$dx_k^{(\alpha)} = \left(\frac{\partial x_k}{\partial X_K} + \Xi_L \frac{\partial \chi_{kL}}{\partial X_K} \right) dX_K + \chi_{kL} d\Xi_L$$

Eq. 15 can be written as

$$\overrightarrow{dx}^{(\alpha)} = (\underline{C} + \underline{\mathscr{C}})\overrightarrow{dX} + \underline{\chi}\overrightarrow{d\Xi} \quad \mathscr{C} = \frac{\partial \chi_{kL}}{\partial X_K}\Xi_L$$



Macrodeformation and microdeformation

Volumen Changes

Material macrovolume element with Ξ fixed

$$dV_o \equiv dX_1 dX_2 dX_3$$

Material minivolume element with X fixed

$$d\mathscr{V}_o \equiv d\Xi_1 d\Xi_2 d\Xi_3$$

After deformation $dV_o \to dv$ and $d\mathscr{V}_o \to d\upsilon$

$$dv = J dX_1 dX_2 dX_3 \quad d\upsilon = j d\Xi_1 d\Xi_2 d\Xi_3$$

Where j and J are the Jacobians of deformation with Ξ and X fixed, respectively

Volumen Changes

$$\overrightarrow{dx}^{(\alpha)} = (\underline{C} + \underline{\mathscr{C}})\overrightarrow{dX} + \underline{\chi}\overrightarrow{d\Xi} \quad \mathscr{C} = \frac{\partial \chi_{kL}}{\partial X_K}\Xi_L$$

Taking Eq. 15

$$J \equiv det\left(\frac{\partial x_k}{\partial X_K} + \frac{\partial \chi_{kM}}{\partial X_K}\Xi_M\right) \quad (57)$$

$$j \equiv det(\chi_{kM}) \quad (58)$$

Applying $det(A) = (det(A \cdot A))^{1/2}$

$$J = \left\{det\left[\left(\frac{\partial x_k}{\partial X_K} + \frac{\partial \chi_{kM}}{\partial X_K}\Xi_M\right)\left(\frac{\partial x_k}{\partial X_L} + \frac{\partial \chi_{kN}}{\partial X_L}\Xi_N\right)\right]\right\}^{1/2}$$

Volumen Changes

$$J = det[\underbrace{\frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L}}_{C_{KL}} + \underbrace{\frac{\partial x_k}{\partial X_K}\frac{\partial \chi_{kN}}{\partial X_L}\Xi_N}_{\Gamma_{KNL}\Xi_N} + \underbrace{\frac{\partial x_k}{\partial X_L}\frac{\partial \chi_{kM}}{\partial X_K}\Xi_M}_{\Gamma_{KML}\Xi_M} + \frac{\partial \chi_{kM}}{\partial X_K}\frac{\partial \chi_{kN}}{\partial X_L}\Xi_M\Xi_N]^{1/2}$$

(59)

Taking the fourth term of the Eq. 59

$$\frac{\partial \chi_{kM}}{\partial X_K}\frac{\partial \chi_{kN}}{\partial X_L}\Xi_M\Xi_N = \underbrace{\frac{\partial \chi_{kM}}{\partial X_K}\frac{\partial x_k}{\partial X_P}}_{\Gamma_{PMK}}\underbrace{\frac{\partial \chi_{kN}}{\partial X_L}\frac{\partial x_k}{\partial X_Q}}_{\Gamma_{QNL}}\Xi_M\Xi_N\underbrace{\frac{\partial X_P}{\partial x_k}\frac{\partial X_Q}{\partial x_k}}_{C^{-1}_{PQ}}$$

The Eq. 59 becomes:

$$J = \left\{ det \left[C_{KL} + \Gamma_{KNL}\Xi_N + \Gamma_{KML}\Xi_M + \underbrace{\Gamma_{PMK}\Gamma_{QNL}\Xi_M\Xi_N C^{-1}_{PQ}}_{\rightarrow 0} \right] \right\}^{1/2} \quad (60)$$

From Eq. 31

$$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL}) \rightarrow C_{KL} = 2E_{KL} + \delta_{KL}$$

From Eq 48

$$\Gamma_{KM} \equiv \Gamma_{KLM}\Xi_L \rightarrow \Gamma_{KNL}\Xi_N = \Gamma_{KML}\Xi_M = \Gamma_{KL}$$

On using this two Eq in 60

$$J = \{det\left[\delta_{KL} + 2E_{KL} + 2\Gamma_{KL}\right]\}^{1/2} \quad (61)$$

Expansion and linearization of this gives

$$J \simeq 1 + E_{KK} + \Gamma_{KK} \quad (62)$$

Volumen Changes

From Eq. 48 $\Gamma_{KM} = \Gamma_{KLM}\Xi_L \rightarrow \Gamma_{KK} = \Gamma_{KRK}\Xi_R$. From Eq. 43 $\Gamma_{KLM} = -e_{KLN}\dfrac{\partial \Phi_N}{\partial X_M} \rightarrow \Gamma_{KRK} = -e_{KRS}\dfrac{\partial \Phi_S}{\partial X_K}$

$$\Gamma_{KK} = -e_{KRS}\frac{\partial \Phi_S}{\partial X_K}\Xi_R = e_{KSR}\frac{\partial \Phi_S}{\partial X_K}\Xi_R = (\nabla \times \overrightarrow{\Phi}) \cdot \Xi_R$$

Using the Eq 62

$$J \simeq 1 + E_{KK} + \Gamma_{KK} = 1 + tr\underline{E} + (\nabla \times \overrightarrow{\Phi}) \cdot \overrightarrow{\Xi}$$

$$\frac{dv}{dVo} - 1 = tr\underline{E} + (\nabla \times \overrightarrow{\Phi}) \cdot \overrightarrow{\Xi} \qquad (63)$$

Volumen Changes

Taking Eq. 58 and calculating the determinant

$$j = \{det\,[\chi_{kK}\chi_{kL}]\}^{1/2} \tag{64}$$

Using the Eq. 23 $\chi_{kK} = \delta_{kK} + \Phi_{kK}$

$$j = \{det\,[(\delta_{kK} + \Phi_{kK})(\delta_{kL} + \Phi_{kL})]\}^{1/2}$$

$$j = \left\{ det \left[\underbrace{\delta_{kK}\delta_{kL}}_{k=K} + \underbrace{\delta_{kK}\Phi_{kL}}_{k=K} + \underbrace{\delta_{kL}\Phi_{kK}}_{k=L} + \underbrace{\Phi_{kK}\Phi_{kL}}_{\to 0} \right] \right\}^{1/2}$$

$$j \approx \{det\,[\delta_{KL} + \Phi_{KL} + \Phi_{LK}]\}^{1/2} = 1 + \Phi_{KK} \tag{65}$$

$\Phi_{KK} = 0$, so that

$$\frac{dv}{d\mathscr{V}_o} - 1 = 0 \tag{66}$$

Some special deformations - Rigid deformation

$\underline{E} = \underline{\mathscr{E}} = \underline{0}, \quad \underline{\Gamma} = \underline{0}$. If $E_{KL} = 0$ implies:

From Eq. 41 $\dfrac{\partial U_K}{\partial X_L} = E_{KL} + R_{KL} \quad \rightarrow \dfrac{\partial U_K}{\partial X_L} = R_{KL}$.

$$\partial U_K = R_{KL} \partial X_L \rightarrow U_K = B_K + R_{KL} X_L \tag{67}$$

If $E_{KL} = 0$ means U is constant, then R_{KL} not depends on X.

If $\Gamma_{KL} = 0 \rightarrow \Gamma_{KLM} = 0 \quad \Gamma_{KM} = \Gamma_{KLM} \underbrace{\Xi_L}_{\neq 0}$

If $\Gamma_{KLM} = 0 \rightarrow \Phi_N$ is constant because $\underbrace{\Gamma_{KLM}}_{=0} = -e_{KLN} \underbrace{\frac{\partial \overbrace{\Phi_N}^{Constant}}{\partial X_M}}_{=0}$

If $\underline{\mathscr{E}} = 0$, from Eq. 42 we have:

$\underbrace{\mathscr{E}_{KL}}_{=0} = \underbrace{E_{KL}}_{=0} + e_{KLM}(R_M - \Phi_M) \rightarrow R_M = \Phi_M \rightarrow R_K = \Phi_K$ where

$R_K = \frac{1}{2} e_{KLM} R_{ML}$; R_{ML} and R_K are independents of X

Macroisochoric deformation: if the material macrovolumen remains unchanged.
Miniisochoric deformation: if the material minivolumen remains unchanged. j is always equal to one.
From Eq. 62:

$$J \simeq 1 + E_{KK} + \Gamma_{KK}$$

$$E_{KK} + \Gamma_{KK} = 0 \tag{68}$$

The Eq. 68 is valid, if E_{KK} and Γ_{KK} are equal to zero.

$$E_{KK} = \nabla \cdot U = 0$$

From Eq. 63 is obtained that $\Gamma_{KK} = (\nabla \times \overrightarrow{\Phi}) \cdot \overrightarrow{\Xi}$. Then $(\nabla \times \overrightarrow{\Phi}) = 0$

Some special deformations - Homogeneous strain

$$\overrightarrow{x}^{(\alpha)} = \overrightarrow{x} + \overrightarrow{\xi} \quad x_k = D_{kK} X_K \quad \xi_k = \mathfrak{D}_{kK} \Xi_K$$

$$\overrightarrow{x}^{(\alpha)} = \underbrace{D_{kK}}_{constant} X_K + \underbrace{\mathfrak{D}_{kK}}_{constant} \Xi_K \quad = \underline{D} \overrightarrow{X} + \underline{\mathfrak{D}} \overrightarrow{\Xi} \tag{69}$$

$x_k = D_{kK} X_K$ is the homogeneous strain in classical continuum
$\xi_k = \mathfrak{D}_{kK} \Xi_K$ is the microhomogeneous deformation.
The Eq. 23 is $\chi_{kK} = \delta_{kK} + \Phi_{kK}$ and the Eq. 37 is
$\Phi_{kK} = -e_{kKM} \Phi_M$. Locating Eq. 37 in Eq. 23 we have:

$$\chi_{kK} = \delta_{kK} - e_{kKM} \Phi_M \quad \chi_{kL} = \delta_{kL} - e_{kLM} \Phi_M$$

$$\underbrace{\chi_{kL} = \delta_{kL} - e_{kLM}\Phi_M}_{\chi_{kL}=\mathfrak{D}_{kL};\Phi_M=\mathfrak{D}_M}$$

$$\mathfrak{D}_{kL} = \delta_{kL} - e_{kLM}\mathfrak{D}_M \tag{70}$$

$$\mathfrak{D}_M = \frac{1}{2}e_{kLM}\mathfrak{D}_{kL} \tag{71}$$

The microhomogeneous deformation can be written thus ($\Phi_M = \mathfrak{D}_M$).

$$\overrightarrow{\xi} = \overrightarrow{\Xi} - \overrightarrow{\Xi} \times \underbrace{\overrightarrow{\mathfrak{D}}}_{constant} \quad (72)$$

The material deformation tensors (Eq. 16, 17 and 18) have the following develop.

Some special deformations - Homogeneous strain

$$C_{KL} \equiv \frac{\partial x_k}{\partial X_K}\frac{\partial x_k}{\partial X_L} \quad x_k = D_{kK}X_K \quad x_k = D_{kL}X_L$$

$$\frac{\partial x_k}{\partial X_K} = \frac{\partial (D_{kK}X_K)}{\partial X_K} = D_{kK}$$

$$\frac{\partial x_k}{\partial X_L} = \frac{\partial (D_{kL}X_L)}{\partial X_L} = D_{kL}$$

$$C_{KL} = D_{kK}D_{kL} \tag{73}$$

$$\Psi_{KL} \equiv \underbrace{\frac{\partial x_k}{\partial X_K}}_{D_{kK}} \underbrace{\chi_{kL}}_{\mathfrak{D}_{kL}} = D_{kK}\mathfrak{D}_{kL} \tag{74}$$

$$\Gamma_{KLM} \equiv \underbrace{\frac{\partial x_k}{\partial X_K}}_{D_{kK}} \frac{\partial \chi_{kL}}{\partial X_M} = D_{kK} \underbrace{\frac{\partial \mathfrak{D}_{kL}}{\partial X_M}}_{=0} = 0 \tag{75}$$

If one use Eq. 70 and Eq. 74, Ψ_{KL} is:

$$\Psi_{KL} = D_{kK}\mathfrak{D}_{kL}$$

$$\Psi_{KL} = D_{kK}(\delta_{kL} - e_{kLM}\mathfrak{D}_{M})$$

$$\Psi_{KL} = \underbrace{D_{kK}\delta_{kL}}_{k=L} - \underbrace{e_{kLM}D_{kK}\mathfrak{D}_{M}}_{k=N}$$

$$\Psi_{KL} = D_{LK} - e_{LMN}D_{NK}\mathfrak{D}_{M} \tag{76}$$

Homogeneous strain - Uniform macrodilatation

$$\underline{D} = \begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}, 0 < D < \infty \tag{77}$$

$$D_{kK} = D\delta_{kK} \quad D_{kL} = D\delta_{kL}$$

Using Eq. 73 we have:

$$C_{KL} = D_{kK}D_{kL} = D\delta_{kK}D\delta_{kL} = D^2\delta_{KL} \tag{78}$$

The macrostretch $(L_{(K)})$ is D $(\sqrt{C_{11}})$. The angle between two vectors of the deformed macroelement is 90^o $(C_{ij} = 0), i \neq j$. The deformation carries a macrocube of unit volume to a macrocube of volumen D^3

On another hand, we have Eq. 74 as:

$$\Psi_{KL} = D_{kK}\mathfrak{D}_{kL} = D\delta_{kK}\mathfrak{D}_{kL} \tag{79}$$

$$\Psi_{KL} = D_{LK} - e_{LMN}D_{NK}\mathfrak{D}_M$$

$$\Psi_{KL} = D\delta_{LK} - e_{LMN}D\delta_{NK}\mathfrak{D}_M = D(\delta_{LK} - \underbrace{e_{LMN}\delta_{NK}\mathfrak{D}_M}_{N=K})$$

$$\Psi_{KL} = D(\delta_{LK} - e_{LMK}\mathfrak{D}_M) = D(\delta_{LK} - e_{KLM}\mathfrak{D}_M) \tag{80}$$

From this is clear:

$$\Psi_{11} = \Psi_{22} = \Psi_{33} = D, \quad \Psi_{KL} = -e_{KLM}D\mathfrak{D}_M, \quad (K \neq L) \tag{81}$$

Homogeneous strain - Uniform macrodilatation

From Eq. 70 we have

$$\mathfrak{D}_{kL} = \delta_{kL} - e_{kLM}\mathfrak{D}_M$$

$$\begin{aligned}
&\mathfrak{D}_{11} = \mathfrak{D}_{22} = \mathfrak{D}_{33} = 1 \\
&\mathfrak{D}_{12} = -e_{123}\mathfrak{D}_3 \quad , \mathfrak{D}_{13} = -e_{132}\mathfrak{D}_2 \quad , \mathfrak{D}_{21} = -e_{213}\mathfrak{D}_3, \\
&\mathfrak{D}_{23} = -e_{231}\mathfrak{D}_1 \quad , \mathfrak{D}_{31} = -e_{312}\mathfrak{D}_2 \quad , \mathfrak{D}_{32} = -e_{321}\mathfrak{D}_1
\end{aligned}$$

$$\mathfrak{D}_{kL} = \begin{bmatrix} 1 & -\mathfrak{D}_3 & \mathfrak{D}_2 \\ \mathfrak{D}_3 & 1 & -\mathfrak{D}_1 \\ -\mathfrak{D}_2 & \mathfrak{D}_1 & 1 \end{bmatrix} \tag{82}$$

Homogeneous strain - Uniform macrodilatation

$\mathfrak{D}_{kL}$ is an orthogonal Tensor for infinitesimal strains. It has the properties that the magnitude of the vectors does not change with application of $\mathfrak{D}_{kL}$. The angle between pair of vectors does not change with $\mathfrak{D}_{kL}$.
$\xi_k = \mathfrak{D}_{kK}\Xi_K$ The microstretch $(l_{(K)})$ (ratio of the magnitude of the deformed microelement to that of the undeformed microelement) is 1.

Homogeneous strain - Uniaxial Strain

$$\underline{D} = \begin{bmatrix} D & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 < D < \infty \tag{83}$$

$C_{KL} = D_{kK} D_{kL}$

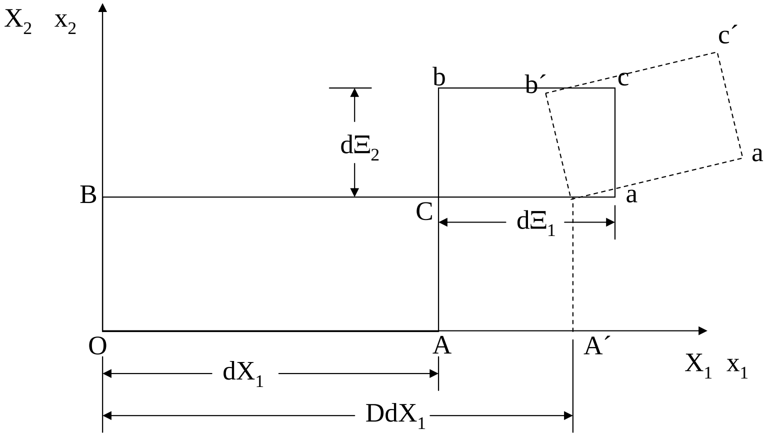
$$\underline{C} = \begin{bmatrix} D & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{C} = \begin{bmatrix} D^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{84}$$

Macrostretch $L_{(K)}$ and Microstretch $l_{(K)}$ are

$$L_{(1)} = D, \quad L_{(2)} = L_{(3)} = 1$$

$$l_{(1)} = l_{(2)} = l_{(3)} = 1$$

Homogeneous strain - Uniaxial Strain



Uniaxial Strain

Homogeneous strain - Simple Shear

$$\underline{D} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -\infty < S < \infty \tag{85}$$

$$\overrightarrow{x}^{(\alpha)} = \underline{D}\overrightarrow{X} + \underline{\mathfrak{D}}\,\overrightarrow{\Xi}$$

$$\begin{bmatrix} x_1^{(\alpha)} \\ x_2^{(\alpha)} \\ x_3^{(\alpha)} \end{bmatrix} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 & -\mathfrak{D}_3 & \mathfrak{D}_2 \\ \mathfrak{D}_3 & 1 & -\mathfrak{D}_1 \\ -\mathfrak{D}_2 & \mathfrak{D}_1 & 1 \end{bmatrix} \begin{bmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{bmatrix}$$

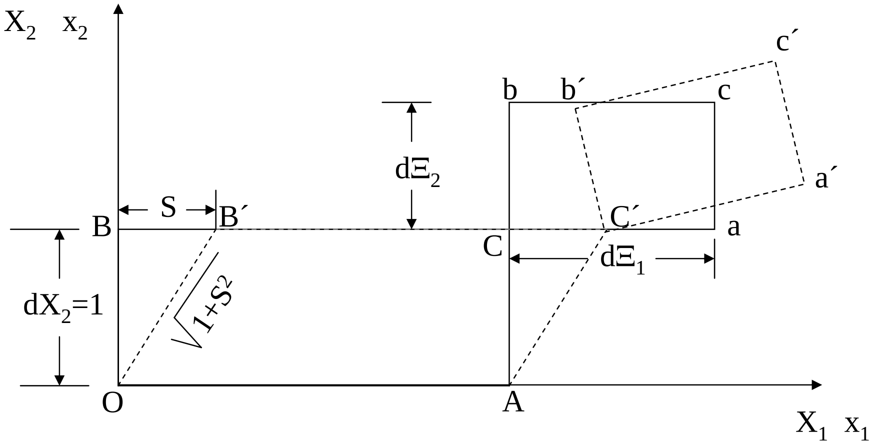
$$\begin{aligned} x_1^{\alpha} &= X_1 + SX_2 + \Xi_1 + \mathfrak{D}_2\Xi_3 - \mathfrak{D}_3\Xi_2 \\ x_2^{\alpha} &= X_2 + \Xi_2 + \mathfrak{D}_3\Xi_1 - \mathfrak{D}_1\Xi_3 \\ x_3^{\alpha} &= X_3 + \Xi_3 + \mathfrak{D}_1\Xi_2 - \mathfrak{D}_2\Xi_1 \end{aligned} \tag{86}$$

Homogeneous strain - Simple Shear

$$C_{KL} = D_{kK} D_{kL}$$

$$\underline{C} = \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{C} = \begin{bmatrix} 1 & S & 0 \\ S & 1+S^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (87)$$

Homogeneous strain - Simple Shear



Simple Shear

Plane Strain

The spatial deformation tensors for plane strain are

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{\Psi} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 \\ \Psi_{21} & \Psi_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (88)$$

$$\Gamma_{12M} = -\Gamma_{21M} = -\frac{\partial \Phi}{\partial X_M} \quad (M = 1, 2), \quad \Gamma_{KLM} = 0$$

Eq. 31 - 33 y 34 provide the relation between the deformation tensors and strains (or displacement vectors)

$$C_{KL} = \delta_{KL} + 2E_{KL} = \delta_{KL} + \frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K}$$

$$\Psi_{KL} = \mathscr{E}_{KL} + \delta_{KL} = \delta_{KL} - e_{KLM}\Phi_M + \frac{\partial U_L}{\partial X_K} \qquad (89)$$

$$\Gamma_{KLM} = -e_{KLN}\frac{\partial \Phi_N}{\partial X_M}$$

Plane Strain

$$\underline{E} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{\mathscr{E}} = \begin{bmatrix} \mathscr{E}_{11} & \mathscr{E}_{12} & 0 \\ \mathscr{E}_{21} & \mathscr{E}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{90}$$

where

$$E_{11} = \frac{\partial U_1}{\partial X_1}, \quad E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}\right), \quad E_{22} = \frac{\partial U_2}{\partial X_2}$$

$$\mathscr{E}_{11} = \frac{\partial U_1}{\partial X_1}, \quad \mathscr{E}_{12} = -\Phi + \frac{\partial U_2}{\partial X_1} \tag{91}$$

$$\mathscr{E}_{21} = \Phi + \frac{\partial U_1}{\partial X_2}, \quad \mathscr{E}_{22} = \frac{\partial U_2}{\partial X_2}$$

Velocity of a material point

Definition: The material derivative of any tensor is defined as the partial derivative of that tensor with respect to time with the material coordinates X_k and Ξ_K held constant.
In order to obtain the relative velocity of a material point $X + \Xi$ with respect to the center of mass X we take the Eq

$$\overrightarrow{\xi} = \underline{\chi}\,\overrightarrow{\Xi} \tag{92}$$

$$\dot{\overrightarrow{\xi}} = \underline{\dot{\chi}}(X,t)\,\overrightarrow{\Xi} \tag{93}$$

$$\Xi_K = \Lambda_{Kk}(x,t)\xi_k \tag{94}$$

Velocity of a material point

On substituting Eq. 94 into 93

$$\overrightarrow{\dot{\xi}} = \underline{\dot{\chi}}\underline{\Lambda}\overrightarrow{\xi} \tag{95}$$

Where

$$\nu_{lk} = \dot{\chi}_{lK}\Lambda_{Kk} \quad \dot{\xi}_l = \nu_{lk}\xi_k \tag{96}$$

The total velocity $\overrightarrow{v}^{(\alpha)}$ of a material point $\overrightarrow{X} + \overrightarrow{\xi}$ can be calculated by

$$\overrightarrow{v}^{(\alpha)} = \overrightarrow{\dot{x}} + \overrightarrow{\dot{\xi}} = \overrightarrow{v} + \underline{\nu}\,\overrightarrow{\xi} \tag{97}$$

From Eq. 38

$$\chi_{kK} = \delta_{kK} - e_{kKM}\Phi_M$$

$$\Lambda_{Kl} = \delta_{Kl} + e_{Klm}\phi_m$$

Velocity of a material point

On replacing this Eq. into Eq. 96:

$$v_{kl} = \dot{\chi}_{kK}\Lambda_{Kl}$$

$$v_{kl} = \left[\frac{\partial(\delta_{kK} - e_{kKM}\Phi_M)}{\partial t}\right][\delta_{Kl} + e_{Klm}\phi_m]$$

$$v_{kl} = -e_{kKM}\dot{\Phi}_M[\delta_{Kl} + e_{Klm}\phi_m]$$

$$v_{kl} = \underbrace{-e_{kKM}\dot{\Phi}_M\delta_{Kl}}_{K=l} - e_{kKM}e_{Klm}\dot{\Phi}_M\phi_m$$

$$v_{kl} = -e_{klM}\dot{\Phi}_M - e_{kKM}e_{Klm}\dot{\Phi}_M\phi_m \quad (98)$$

For the linear theory, this is simplified to

$$v_{kl} \simeq -e_{klm}\dot{\phi}_m \quad (99)$$

Velocity of a material point

On introducing an axial vector ν_k, called microgyration vector

$$\nu_k = \frac{1}{2} e_{klm} \nu_{ml} \quad \nu_{kl} = -e_{klm} \nu_m \tag{100}$$

we see that

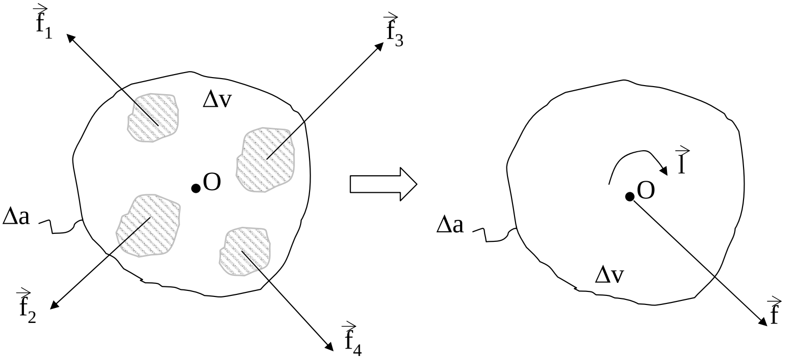
$$\nu_k = \dot{\phi}_k \tag{101}$$

The Eq. $\overrightarrow{\dot{\xi}} = \underline{\nu} \vec{\xi}$ now reads

$$\begin{aligned} \dot{\xi}_l &= \nu_{lk} \xi_k & (102) \\ \dot{\xi}_l &= -e_{lkm} \nu_m \xi_k & (103) \\ \dot{\xi}_l &= e_{mkl} \nu_m \xi_k & (104) \\ \overrightarrow{\dot{\xi}} &= \vec{\nu} \times \vec{\xi} & (105) \\ \overrightarrow{\dot{\xi}} &= -\vec{\xi} \times \vec{\nu} & (106) \end{aligned}$$

Loads

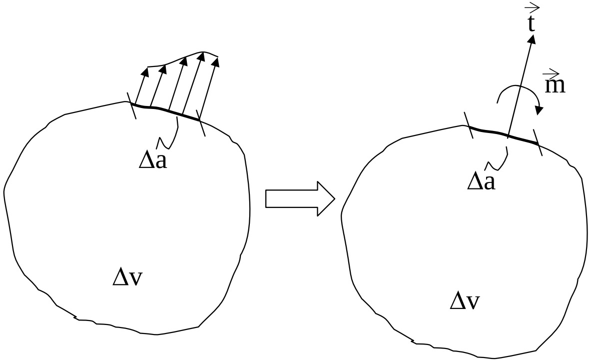
Body loads



Macrovolume element with force and couple body

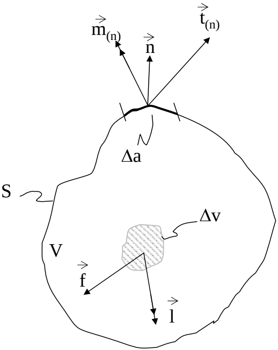
$\vec{f}$ = Body force per unit mass; $\vec{l}$ = Body couple per unit mass

Surface loads



Macrovolume element with surface loads

Surface and body loads



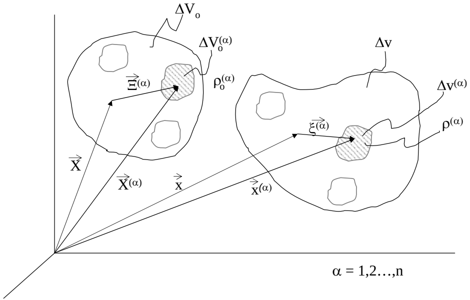
Surface and body loads on volumen

$$\mathfrak{F} = \oint_s \overrightarrow{t}_{(n)} da + \int_v \rho \overrightarrow{f} dv$$
$$\mathfrak{M} = \oint_s [\overrightarrow{m}_{(n)} + \overrightarrow{x} \times \overrightarrow{t}_{(n)}] da + \int_v \rho (\overrightarrow{l} + \overrightarrow{x} \times \overrightarrow{f}) dv$$

Mechanical balance laws

1. Conservation of mass
2. Balance of momentum
3. Balance of momentum of momentum

The total mass of each microelement remains constant during any deformation.



Macromass elements containing micromass elements

Conservation of mass

$$\rho_o^{(\alpha)} \Delta V_o^{(\alpha)} = \rho^{(\alpha)} \Delta v^{(\alpha)} \tag{107}$$

The total mass of a macrovolume before and after deformation, is given by

$$\rho_o \Delta V_o \equiv \sum_{\alpha=1}^{N} \rho_o^{(\alpha)} \Delta V_o^{(\alpha)} \tag{108}$$

$$\rho \Delta v \equiv \sum_{\alpha=1}^{N} \rho^{(\alpha)} \Delta v^{(\alpha)} \tag{109}$$

In view of Eq 107 we see that

$$\rho_o \Delta V_o = \rho \Delta v \tag{110}$$

Conservation of mass

If ΔV_o and $\Delta v \to dV_o$ and dv then

$$\rho_o dV_o = \rho dv \quad \frac{\rho_o}{\rho} = \frac{dv}{dV_o} \equiv J = det(\frac{\partial x_k}{\partial X_K}) \qquad (111)$$

Eq. 111 are equivalents expressions of the principle of conservation of mass for the macrovolume element.

X is the position of the center of mass of a macroelement. Accordingly,

$$\sum_{\alpha} \rho_o^{(\alpha)} \Xi^{(\alpha)} \Delta V_o^{(\alpha)} = \overrightarrow{0}$$

Conservation of mass

Using Eq. 107 and 9, this gives

$$\overrightarrow{\Xi}^{(\alpha)} = \underline{\Lambda}\,\overrightarrow{\xi}^{(\alpha)}$$

$$\underbrace{\underline{\Lambda}}_{\neq 0} \sum_{\alpha} \rho^{(\alpha)} \xi^{(\alpha)} \Delta v^{(\alpha)} = \overrightarrow{0} \tag{112}$$

This shows that the position vector x is the center of mass of the deformed macrovolume.
Global balance equation for mass is obtained by integrating Eq. 110

$$\int_V \rho_o dV_o = \int_v \rho dv \tag{113}$$

Principle of balance of momentum

The time rate of change of momentum is equal to the sum of all forces acting on a body

$$\begin{aligned}
\Delta p = \sum_\alpha \rho^{(\alpha)} \mathbf{v}^{(\alpha)} \Delta v^{(\alpha)} &= \sum_\alpha \rho^{\alpha} (\mathbf{v} + \dot{\xi}) \Delta v^{(\alpha)} \\
&= \sum_\alpha \rho^{(\alpha)} (\mathbf{v} - \xi \times \nu) \Delta v^{(\alpha)} \\
&= \sum_\alpha \rho^{(\alpha)} \mathbf{v} \Delta v^{(\alpha)} - \sum_\alpha \rho^{(\alpha)} (\xi \times \nu) \Delta v^{(\alpha)} \\
&= \mathbf{v} \sum_\alpha \rho^{(\alpha)} \Delta v^{(\alpha)} + \nu \times \underbrace{\sum_\alpha \rho^{(\alpha)} \xi \Delta v^{(\alpha)}}_{=0}
\end{aligned}$$

Principle of balance of momentum

With Eq. 109, in the limit we can write

$$dp = \rho \mathbf{v} dv$$

The total momentum of the body is

$$p = \int_v \rho \mathbf{v} dv \tag{114}$$

The principle of balance of momentum is expressed by

$$\frac{d}{dt} \int_v \rho \mathbf{v} dv = \oint_s t_{(n)} da + \int_v \rho f dv \tag{115}$$

where
$t_{(n)}$ is the surface traction per unit area acting on the surface of the body with normal n.
Eq. 115 is the same given in classical continuum mechanics.

Principle of balance of moment of momentum

The time rate of change of moment of momentum about a point is equal to the sum of all couples and the moment of all forces about that point.

The mechanical moment of momentum of a microelement is

$$\overrightarrow{x}^{(\alpha)} \times \rho^{(\alpha)} \mathbf{v}^{(\alpha)} \Delta v^{(\alpha)}$$

The total moment of momentum of a macroelement is calculated by:

$$\Delta \mathfrak{M} = \sum_{\alpha} \overrightarrow{x}^{(\alpha)} \times \rho^{(\alpha)} \mathbf{v}^{(\alpha)} \Delta v^{(\alpha)} \tag{116}$$

$$\Delta \mathfrak{M} = \sum_{\alpha} (\overrightarrow{x} + \overrightarrow{\xi}) \times \rho^{(\alpha)} (\mathbf{v} + \overrightarrow{\dot{\xi}}) \Delta v^{(\alpha)} \tag{117}$$

$$\begin{aligned}\Delta\mathfrak{M} &= \vec{x} \times \mathbf{v} \sum_{\alpha} \rho^{(\alpha)} \Delta v^{(\alpha)} + \sum_{\alpha} \vec{\xi} \times \rho^{(\alpha)} \dot{\vec{\xi}} \Delta v^{(\alpha)} \\ &+ \underbrace{\vec{x} \times \sum_{\alpha} \rho^{(\alpha)} \dot{\vec{\xi}} \Delta v^{(\alpha)}}_{=0} - \underbrace{\mathbf{v} \times \sum_{\alpha} \rho^{(\alpha)} \vec{\xi} \Delta v^{(\alpha)}}_{=0}\end{aligned}$$

$$\Delta\mathfrak{M} = \vec{x} \times \mathbf{v} \sum_{\alpha} \rho^{(\alpha)} \Delta v^{(\alpha)} + \underbrace{\sum_{\alpha} \vec{\xi} \times \rho^{(\alpha)} (\vec{v} \times \vec{\xi}) \Delta v^{(\alpha)}}_{\rho\sigma\Delta v}$$

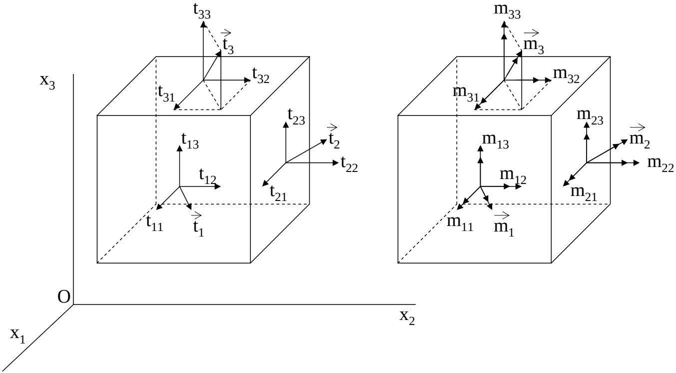
$$\mathfrak{M} = \int_{v} (\vec{x} \times \rho\mathbf{v} + \rho\sigma) dv \qquad (118)$$

Principle of balance of moment of momentum

The principle of moment of momentum is expressed by

$$\frac{d}{dt}\int_{v}(\overrightarrow{x}\times\rho\mathbf{v}+\rho\sigma)dv=\oint_{s}(\overrightarrow{x}\times t_{(n)}+m_{(n)})da+\int_{v}\rho(l+\overrightarrow{x}\times\overrightarrow{f})dv \tag{119}$$

Stress and couple stress



Stress and couple stress

Stress and couple stress

$$\underline{t} = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}, \quad \underline{m} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{yx} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{bmatrix} \tag{120}$$

Local balance laws

$$\int_V \rho_o dV_o = \int_v \rho dv; \quad J = \frac{dv}{dV_o} \tag{121}$$

$$\int_v \rho_o \frac{dv}{J} = \int_v \rho dv; \quad \int_v \rho_o dv = \int_v \rho J dv$$

$$\int_v (\rho_o - \rho J) dv = 0; \quad \rho_o - \rho J = 0 \tag{122}$$

Equation of local mass conservation

$$\frac{\rho_o}{\rho} = J \tag{123}$$

Local balance laws

From the principle of balance of momentum (Eq. 115):

$$\frac{d}{dt}\int_v \rho \mathbf{v} dv = \oint_s \overrightarrow{t}_{(n)} da + \int_v \rho \overrightarrow{f} dv$$

$$\int_v \rho \overrightarrow{a} dv = \oint_s \overrightarrow{t}_{(n)} da + \int_v \rho \overrightarrow{f} dv \tag{124}$$

$$\int_v \rho \overrightarrow{a} dv = \oint_s \underline{t}^T n da + \int_v \rho \overrightarrow{f} dv \tag{125}$$

$$\int_v \rho \overrightarrow{a} dv = \oint_s n \underline{t} da + \int_v \rho \overrightarrow{f} dv \tag{126}$$

$$\int_v \rho \overrightarrow{a} dv = \int_v \nabla \underline{t} dv + \int_v \rho \overrightarrow{f} dv \tag{127}$$

$$\int_v \rho \overrightarrow{a} dv = \int_v Div \underline{t} dv + \int_v \rho \overrightarrow{f} dv \tag{128}$$

Local balance laws

$$\int_v [\underbrace{Div\underline{t} + \rho(\overrightarrow{f} - \overrightarrow{a})}_{=0}]dv = 0 \tag{129}$$

$$\frac{\partial t_{lk}}{\partial X_l} + \rho(f_k - a_k) = 0 \tag{130}$$

Local balance laws

From principle of moment of momentum (119)

$$\frac{d}{dt}\int_v (\overrightarrow{x} \times \rho \overrightarrow{\mathbf{v}} + \rho \overrightarrow{\sigma}) dv = \oint_s (\overrightarrow{x} \times \overrightarrow{t_{(n)}} + \overrightarrow{m_{(n)}}) da + \int_v \rho (\overrightarrow{l} + \overrightarrow{x} \times \overrightarrow{f}) dv \quad (131)$$

$$\int_v (\underbrace{\overrightarrow{\mathbf{v}} \times \rho \overrightarrow{\mathbf{v}}}_{=0} + \overrightarrow{x} \times \rho \underbrace{\frac{d\overrightarrow{\mathbf{v}}}{dt}}_{\overrightarrow{a}} + \rho \dot{\overrightarrow{\sigma}}) dv = \oint_s (\overrightarrow{x} \times \overrightarrow{t_{(n)}} + \overrightarrow{m_{(n)}}) da +$$

$$\int_v \rho (\overrightarrow{l} + \overrightarrow{x} \times \overrightarrow{f}) dv \quad (132)$$

Local balance laws

$$\int_v \rho(\overrightarrow{x} \times \overrightarrow{a} + \dot{\overrightarrow{\sigma}})dv = \underbrace{\oint_s (\overrightarrow{x} \times \underline{t}^T \overrightarrow{n})da}_{1} + \oint_s \underline{m}^T \overrightarrow{n} da +$$

$\int_v \rho(\overrightarrow{l} + \overrightarrow{x} \times \overrightarrow{f})dv$(133)

$$\int_v \rho(\overrightarrow{x} \times \overrightarrow{a} + \dot{\overrightarrow{\sigma}})dv = \underbrace{\oint_s (\overrightarrow{x} \times \overrightarrow{n} \underline{t})da}_{1} + \int_v \nabla \underline{m} dv +$$

$\int_v \rho(\overrightarrow{l} + \overrightarrow{x} \times \overrightarrow{f})dv$(134)

Organizing terms

$$\int_v [\nabla \underline{m} + \rho(\overrightarrow{l} - \dot{\overrightarrow{\sigma}})]dv + \int_v \overrightarrow{x} \times [\rho(\overrightarrow{f} - \overrightarrow{a})]dv + \underbrace{\oint_s (\overrightarrow{x} \times \overrightarrow{n}\,\underline{t})da}_{1} = 0 \tag{135}$$

In components, Term 1 is:

$$\oint_s (e_{ijk}x_j n_l t_{lk})da = \oint_s n_l(e_{ijk}x_j t_{lk})da = \int_v \frac{\partial(e_{ijk}x_j t_{lk})}{\partial x_l}dv \tag{136}$$

$$= \int_v [e_{ijk}\underbrace{\frac{\partial x_j}{\partial x_l}}_{\delta_{jl}} t_{lk} + e_{ijk}\frac{\partial t_{lk}}{\partial x_l}x_j]dv \tag{137}$$

Local balance laws

$$= \underbrace{\int_v e_{ijk}\delta_{jl}t_{lk}dv}_{j=l} + \underbrace{\int_v e_{ijk}x_j \frac{\partial t_{lk}}{\partial x_l}dv}_{\overrightarrow{x}\times\frac{\partial t_{lk}}{\partial x_l}} \quad (138)$$

$$\oint_s (e_{ijk}x_j n_l t_{lk})da = \int_v e_{ilk}t_{lk}dv + \int_v \overrightarrow{x} \times Div(\underline{t})dv \quad (139)$$

On substituting into Eq 135 results:

$$\int_v [\nabla \underline{m} + \rho(\overrightarrow{l} - \overrightarrow{\dot{\sigma}})]dv + \int_v \overrightarrow{x} \times [\rho(\overrightarrow{f} - \overrightarrow{a})]dv +$$

$\int_v e_{ilk}t_{lk}dv + \int_v \overrightarrow{x} \times Div(\underline{t})dv = 0$(140)

Local balance laws

$$\int_v \left[\frac{\partial m_{lk}}{\partial x_l} + \rho(\overrightarrow{l} - \dot{\overrightarrow{\sigma}}) \right] dv + \underbrace{\int_v \overrightarrow{x} \times [\rho(\overrightarrow{f} - \overrightarrow{a}) + Div\underline{t}] dv}_{=0} + \int_v e_{ilk} t_{lk} dv = \overrightarrow{0} \quad (141)$$

In components

$$\frac{\partial m_{lk}}{\partial x_l} + \rho(l_k - \dot{\sigma}_k) + e_{ilk} t_{lk} = 0 \quad (142)$$

In rectangular coordinates the expanded expressions of Eqs. 130 and 142 are:

$$\frac{\partial t_{lk}}{\partial x_l} + \rho(f_k - a_k) = \overrightarrow{0}$$

Local balance laws

$$
\begin{aligned}
\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} + \frac{\partial t_{zx}}{\partial z} + \rho(f_x - a_x) &= 0 \\
\frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{zy}}{\partial z} + \rho(f_y - a_y) &= 0 \\
\frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + \rho(f_z - a_z) &= 0
\end{aligned} \tag{143}
$$

$$
\begin{aligned}
\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} + \frac{\partial m_{zx}}{\partial z} + t_{yz} - t_{zy} + \rho(l_x - \dot{\sigma}_x) &= 0 \\
\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{zy}}{\partial z} + t_{zx} - t_{xz} + \rho(l_y - \dot{\sigma}_y) &= 0 \\
\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \frac{\partial m_{zz}}{\partial z} + t_{xy} - t_{yx} + \rho(l_z - \dot{\sigma}_z) &= 0
\end{aligned} \tag{144}
$$

In linear micropolar elasticity, the strain measures are

$$\mathscr{E}_{KL} = E_{KL} + e_{KLM}(R_M - \Phi_M) \quad (145)$$

$$\Gamma_{KLM} = e_{KLN}\frac{\partial \Phi_N}{\partial M} \quad (146)$$

Only 9 components $\frac{\partial \Phi_N}{\partial X_M}$ are independent and non zero (Eq. 53). We may, instead of Γ_{KLM}, to use the axial tensor $\frac{\partial \Phi_K}{\partial X_L}$.
For the anisotropic micropolar elastic solids, we have the following relations.

$$t_{KL} = A_{KLMN}\mathscr{E}_{MN} \quad (147)$$

$$m_{KL} = B_{KLMN}\frac{\partial \Phi_M}{\partial X_N} \quad (148)$$

Theory of micropolar elasticity

If the body is isotropic, the tensors A and B have to be isotropics. The most general form of second order isotropic tensor is:

$$A_{KLMN} = A_1\delta_{KL}\delta_{MN} + A_2\delta_{KM}\delta_{LN} + A_3\delta_{KN}\delta_{LM} \quad (149)$$

$$B_{KLMN} = B_1\delta_{KL}\delta_{MN} + B_2\delta_{KM}\delta_{LN} + B_3\delta_{KN}\delta_{LM} \quad (150)$$

The Eq 147 takes the following form

$$t_{KL} = \underbrace{(A_1\delta_{KL}\delta_{MN} + A_2\delta_{KM}\delta_{LN} + A_3\delta_{KN}\delta_{LM})}_{A_{KLMN}}\mathscr{E}_{MN}$$

$$t_{KL} = A_1\delta_{KL}\underbrace{\delta_{MN}\mathscr{E}_{MN}}_{M=N} + A_2\delta_{KM}\underbrace{\delta_{LN}\mathscr{E}_{MN}}_{N=L} + A_3\delta_{KN}\underbrace{\delta_{LM}\mathscr{E}_{MN}}_{M=L}$$

$$t_{KL} = A_1 \delta_{KL} \mathscr{E}_{MM} + A_2 \underbrace{\delta_{KM} \mathscr{E}_{ML}}_{M=K} + A_3 \underbrace{\delta_{KN} \mathscr{E}_{LN}}_{N=K}$$

$$t_{KL} = A_1 \delta_{KL} \mathscr{E}_{MM} + A_2 \mathscr{E}_{KL} + A_3 \mathscr{E}_{LK} \quad (151)$$

Eq 148 is modified thus:

$$m_{KL} = \underbrace{(B_1 \delta_{KL} \delta_{MN} + B_2 \delta_{KM} \delta_{LN} + B_3 \delta_{KN} \delta_{LM})}_{B_{KLMN}} \frac{\partial \Phi_M}{\partial X_N}$$

$$m_{KL} = B_1 \delta_{KL} \underbrace{\delta_{MN} \frac{\partial \Phi_M}{\partial X_N}}_{M=N} + B_2 \delta_{KM} \underbrace{\delta_{LN} \frac{\partial \Phi_M}{\partial X_N}}_{N=L} + B_3 \delta_{KN} \underbrace{\delta_{LM} \frac{\partial \Phi_M}{\partial X_N}}_{M=L}$$

$$m_{KL} = B_1 \delta_{KL} \frac{\partial \Phi_M}{\partial X_M} + B_2 \underbrace{\delta_{KM} \frac{\partial \Phi_M}{\partial X_L}}_{M=K} + B_3 \underbrace{\delta_{KN} \frac{\partial \Phi_L}{\partial X_N}}_{N=K}$$

$$m_{KL} = B_1 \delta_{KL} \frac{\partial \Phi_R}{\partial X_R} + B_2 \frac{\partial \Phi_K}{\partial X_L} + B_3 \frac{\partial \Phi_L}{\partial X_L} \tag{152}$$

On introducing

$$A_1 = \lambda, \quad A_2 = \mu + \kappa, \quad A_3 = \mu \tag{153}$$

$$B_1 = \alpha, \quad B_2 = \beta, \quad B_3 = \gamma \tag{154}$$

The eqs. 151 and 152 can be written as:

$$t_{KL} = \lambda \delta_{KL} \mathscr{E}_{MM} + (\mu + \kappa) \mathscr{E}_{KL} + \mu \mathscr{E}_{LK} \tag{155}$$

$$m_{KL} = \alpha \delta_{KL} \frac{\partial \Phi_R}{\partial X_R} + \beta \frac{\partial \Phi_K}{\partial X_L} + \gamma \frac{\partial \Phi_L}{\partial X_K} \tag{156}$$

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The Eq. 155 can be written alternatively as:

$$\mathscr{E}_{MM} = E_{RR}$$

$$t_{KL} = \lambda \delta_{KL} E_{RR} + \mu(\mathscr{E}_{KL} + \mathscr{E}_{LK}) + \kappa \mathscr{E}_{KL}$$

Eq. 145 is expressed by:

$$\mathscr{E}_{KL} = E_{KL} + e_{KLM}(R_M - \Phi_M) \quad \mathscr{E}_{LK} = E_{LK} + e_{LKM}(R_M - \Phi_M)$$

Substituting in the last equation

$$t_{KL} = \lambda \delta_{KL} E_{RR} + \mu(E_{KL} + e_{KLM}(R_M - \Phi_M) + E_{LK} + e_{LKM}(R_M - \Phi_M)) +$$

$$\kappa(E_{KL} + e_{KLM}(R_M - \Phi_M))$$

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$$t_{KL} = \lambda \delta_{KL} E_{RR} + \mu (E_{KL} + e_{KLM}(R_M - \Phi_M) + E_{LK} - e_{KLM}(R_M - \Phi_M)) +$$
$$\kappa (E_{KL} + e_{KLM}(R_M - \Phi_M))$$

$$t_{KL} = \lambda \delta_{KL} E_{RR} + \mu (E_{KL} + E_{LK}) + \kappa (E_{KL} + e_{KLM}(R_M - \Phi_M))$$

$$t_{KL} = \lambda \delta_{KL} E_{RR} + 2\mu E_{KL} + \kappa (E_{KL} + e_{KLM}(R_M - \Phi_M))$$

$$t_{KL} = \lambda \delta_{KL} E_{RR} + (2\mu + \kappa) E_{KL} + \kappa e_{KLM}(R_M - \Phi_M) \qquad (157)$$

Theory of micropolar elasticity

$$t_{KL} = \lambda \delta_{KL} E_{RR} + (2\mu + \kappa) E_{KL} + \kappa e_{KLM} (R_M - \Phi_M)$$

$$m_{KL} = \alpha \delta_{KL} \frac{\partial \Phi_R}{\partial X_R} + \beta \frac{\partial \Phi_K}{\partial X_L} + \gamma \frac{\partial \Phi_L}{\partial X_K}$$

There are 4 extra modulus in micropolar isotropic elasticity $\alpha, \beta, \gamma, \kappa$. If these modulus are set equal to zero, it is obtained the Hooke isotropic elasticity.